140. *Semigroups Whose Arbitrary Subsets Containing a Definite Element are Subsemigroups*

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1. Consider a semigroup S satisfying the following condition: Any subset of S which contains a definite element e is a subsemigroup of S.

A semigroup S is called a β^*-semigroup if S satisfies the above condition.

For example semigroups of order 2, β-semigroups [4][1] and Rédei's semigroups are all β^*-semigroups, where by a Rédei's semigroup we mean a semigroup satisfying the condition that any non-empty subset is a subsemigroup [2].[2]

2. Immediately we have that a homomorphic image of S is a β^*-semigroup and any subset of S which contains e is also a β^*-semigroup.

Putting now $T=\{x\in S;\ x^2=x\}$, $U=\{x\in S;\ x^2=e,\ x\neq e,\ ex=xe=e\}$, and $V=\{x\in S;\ x^2=e,\ x\neq e,\ ex=xe=x\}$, it follows that V has at most one element and $S=T+U+V$ (disjoint class-sum).

We define a relation $\approx$ as follows:

$a\approx b$ means that at least one of $a\underset{l}{\sim}b$, $a\underset{r}{\sim}b$ and $a\sim b$ holds, provided that $a\underset{l}{\sim}b$ [$a\underset{r}{\sim}b$] means $ab=a$ and $ba=b$ [$ab=b$ and $ba=a$] for a, b in T, $a\sim b$ does $ab=ba=e$ for a, b in $S\setminus T$.[3]

Then we have the following lemmas.

Lemma 1. $\approx$ is an equivalence relation defined in S.

Lemma 2. For any a, b in U, any c in T and w in V $a\approx b$, $w\not\approx a$ ($\not\approx$ denotes the negation of $\approx$), $w\not\approx c$ and $a\not\approx c$.

Lemma 3. If $V\neq\square$,[4] then $e\approx a$ implies $e=a$.

Thus we have

Theorem 1. S can be represented as

$$S=\sum_{\alpha\in\Lambda}S_\alpha=\sum_{\lambda\in\Lambda_l}S_\lambda+\sum_{\mu\in\Lambda_r}S_\mu+\sum_{\nu\in\Lambda_0}S_\nu \text{ (disjoint class-sum)}$$

where $\Lambda=\Lambda_l\cup\Lambda_r\cup\Lambda_0$, $\Lambda_0=\{\omega,\varepsilon,\upsilon\}$,

S_λ, $\lambda\in\Lambda_l$ [S_μ, $\mu\in\Lambda_r$] is a maximal left [right] zero[5] subsemigroup which contains no e,

1) The numbers in brackets refer to the references at the end of the paper.

2) See Theorem 50 in [2].

3) $S\setminus T$ means the set of all elements belonging to S but not to T.

4) $\square$ denotes the empty set.

5) A left [right] zero is a semigroup defined by $xy=x$ [$xy=y$] for all x, y.

S_ε is a maximal left or right zero subsemigroup which contains e and especially $S_\varepsilon=\{e\}$ when $S_\omega \neq \Box$,

$S_\omega=\Box$ or $\{w\}$,

$S_\upsilon=U$.

3. Next, we define an ordering $a \gtrsim b$ meaning $a \geqq b$, or $a \underset{l}{\gtrsim} b$, or $a \underset{r}{\gtrsim} b$, defined as follows:

$a \geqq b$ means either $a=b$ or $ab=ba=a$,

$a \underset{l}{\gtrsim} b$ $[a \underset{r}{\gtrsim} b]$ does $ab=a$ and $ba=e$ $[ab=e$ and $ba=a]$ for $a \neq e$, $b \neq e$, $a \neq b$.

It is easily shown that

Lemma 4. $\gtrsim$ is a partial ordering defined in S.

Lemma 5. For $w \in S_\omega$, $u \in S_\upsilon$, and e

$w \gtrsim x \gtrsim e \gtrsim y \gtrsim u$ implies $x=e$ and $y=e$.

Define $a \not\gtrless b$ meaning that $ab=ba=e$ for $a \neq e$, $b \neq e$, $a \neq b$, and $a \nsim b$ ($\nsim$ denotes the negation of $\sim$). Then we have

Lemma 6. Let $a \not\gtrless b$. Then

(i) $e>a$ ($>$ denotes that $\geqq$ and $\neq$) and $e>b$,

(ii) if there exists c ($\leqq e$) such that $c>a$ and $c>b$, then $c=e$.

Lemma 7. Let $b \approx c$, $b \neq c$, and $a \not\approx b$. Then

(i) $a>b$ implies $a>c$,

(ii) $a \underset{l}{\gtrsim} b$ $[a \underset{r}{\gtrsim} b]$ implies $a \underset{l}{\gtrsim} c$ $[a \underset{r}{\gtrsim} c]$,

(iii) $a \not\gtrless b$ implies $a \not\gtrless c$ if $b \nsim c$,

$a \not\gtrsim c$[6] if $b \sim c$.

Lemma 8. Let $b \underset{l}{\sim} c$ $[b \underset{r}{\sim} c]$, $b \neq c$, and $a \not\approx b$. Then $b \underset{r}{\gtrsim} a$ $[b \underset{l}{\gtrsim} a]$ does not occur and

(i) $b>a$ implies $c>a$ or $c \underset{l}{\gtrsim} a$ $[c \underset{r}{\gtrsim} a]$ if $e \approx b$,

$c>a$ if $e \not\approx b$,

(ii) $b \underset{l}{\gtrsim} a$ $[b \underset{r}{\gtrsim} a]$ implies $e \approx b$ and $c>a$ or $c \underset{l}{\gtrsim} a$ $[c \underset{r}{\gtrsim} a]$.

Lemma 9. Let $b \sim c$, $b \neq c$ and $a \not\approx b$. Then

$b \gtrsim a$ implies $a \not\gtrsim c$.

Let $\overline{S}=\{S_\alpha\}_{\alpha \in \Lambda}$ and define $\geqq$ and $\not\gtrless$ in $\overline{S}$ as follows:

$S_\alpha \geqq S_\beta$ means $S_\alpha=S_\beta$ or $x>y$ for every $x \in S_\alpha$ and every $y \in S_\beta$,

$S_\alpha \not\gtrless S_\beta$ does $x \not\gtrless y$ for every $x \in S_\alpha$ and every $y \in S_\beta$.

By $S_\alpha > S_\beta$ we denotes that $S_\alpha \geqq S_\beta$ and $S_\alpha \neq S_\beta$.

And, we define $\geqq$ ($>$ or $=$) and $\not\gtrless$ in Λ as follows:

$\alpha>\beta$ means $S_\alpha>S_\beta$, $\alpha=\beta$ does $S_\alpha=S_\beta$ and $\alpha \not\gtrless \beta$ does $S_\alpha \not\gtrless S_\beta$.

Then it is easily shown that $\overline{S}$ is order isomorphic onto Λ under a mapping $S_\alpha \to \alpha$. And we have

Theorem 2. Λ is a partially ordered set with respect to $\geqq$ which contains a definite element ε and has the following properties:

6) $\not\gtrsim$ denotes the negation of $\gtrsim$.

(i) If $\omega \in \Lambda$, then for any $\alpha \in \Lambda$ one and only one of $\alpha > \varepsilon$, $\alpha = \varepsilon$ and $\varepsilon > \alpha$ holds.

(ii) $\alpha > \varepsilon$ implies $\alpha > \upsilon$ and $\alpha > \upsilon$ implies $\alpha \geqq \varepsilon$.

(iii) For any $\alpha(\neq \varepsilon, \neq \upsilon)$, $\beta(\neq \varepsilon, \neq \upsilon)$ in Λ one and only one of $\alpha > \beta$, $\alpha = \beta$, $\beta > \alpha$ and $\alpha \not\gtrless \beta$ holds.

(iv) Unless $\alpha \not\geqq \varepsilon$ and $\beta \not\geqq \varepsilon$, then one and only one of $\alpha > \beta$, $\alpha = \beta$ and $\beta > \alpha$ holds.

(v) If there exists $\gamma(\not\geqq \varepsilon)$ in Λ such that $\gamma \geqq \alpha$ and $\gamma \geqq \beta$, then one and only one of $\alpha > \beta$, $\alpha = \beta$ and $\beta > \alpha$ holds.

4. Put $\overline{S}_\varepsilon(\alpha) = \{y \in S_\varepsilon;\ y \geqq x\}$, $S_\varepsilon^l(\alpha) = \{y \in S_\varepsilon;\ y \underset{l}{\gtrsim} x\}$, $S_\varepsilon^r(\alpha) = \{y \in S_\varepsilon;\ y \underset{r}{\gtrsim} x\}$, $\overline{S}_\upsilon(\alpha) = \{z \in S_\upsilon;\ z \geqq x\}$, $S_\upsilon^l(\alpha) = \{z \in S_\upsilon;\ z \underset{l}{\gtrsim} x\}$, $S_\upsilon^r(\alpha) = \{z \in S_\upsilon;\ z \underset{r}{\gtrsim} x\}$ and $\widetilde{S}_\upsilon(\alpha) = \{z \in S_\upsilon;\ z \not\gtrless x\}$ for a fixed element x of S_α.

Then $\overline{S}_\varepsilon(\alpha)$, $S_\varepsilon^l(\alpha)$, $S_\varepsilon^r(\alpha)$ are defined for all $\alpha(\not\geqq \varepsilon)$ in Λ and $\overline{S}_\upsilon(\alpha)$, $S_\upsilon^l(\alpha)$, $S_\upsilon^r(\alpha)$ and $\widetilde{S}_\upsilon(\alpha)$ are done for all α in Λ such that $\alpha \not\geqq \varepsilon$ and $\alpha \neq \upsilon$.

And these all subsets are determined uniquely by α and are mutually disjoint.

And we have

Theorem 3. (i) $\overline{S}_\varepsilon(\alpha) \ni e$ for every $\alpha(\not\geqq \varepsilon)$ in Λ, and especially $\overline{S}_\varepsilon(\upsilon) = \{e\}$.

(ii) For every $\alpha(\not\geqq \varepsilon)$ in Λ

$$S_\varepsilon = \overline{S}_\varepsilon(\alpha) + S_\varepsilon^l(\alpha) \quad \text{if } S_\varepsilon \text{ is a left zero,}$$
$$= \overline{S}_\varepsilon(\alpha) + S_\varepsilon^r(\alpha) \quad \text{if } S_\varepsilon \text{ is a right zero}$$

and for every $\alpha(\not\geqq \varepsilon, \neq \upsilon)$ in Λ

$$S_\upsilon = \overline{S}_\upsilon(\alpha) + S_\upsilon^l(\alpha) + S_\upsilon^r(\alpha) + \widetilde{S}_\upsilon(\alpha).$$

(iii) For every x in $S_\alpha(\alpha \not\geqq \varepsilon)$ it follows that

$$y \geqq x \text{ for every } y \in \overline{S}_\varepsilon(\alpha),$$
$$y \underset{l}{\gtrsim} x \ [y \underset{r}{\gtrsim} x] \text{ for every } y \in S_\varepsilon^l(\alpha)\ [y \in S_\varepsilon^r(\alpha)]$$

and for every x in S_α $(\alpha \not\geqq \varepsilon, \neq v)$ it follows

$$z \geqq x \text{ for every } z \in \overline{S}_\upsilon(\alpha),$$
$$z \underset{l}{\gtrsim} x \ [z \underset{r}{\gtrsim} x] \text{ for every } z \in S_\upsilon^l(\alpha)\ [z \in S_\upsilon^r(\alpha)],$$
$$z \not\gtrless x \text{ for every } z \in \widetilde{S}_\upsilon(\alpha).$$

(iv) For $\alpha(\not\geqq \varepsilon, \neq \upsilon)$, $\beta(\not\geqq \varepsilon, \neq \upsilon)$ it follows that if $\alpha > \beta$, then

1) $\overline{S}_\varepsilon(\alpha) \subseteq \overline{S}_\varepsilon(\beta)$,

2) $\overline{S}_\upsilon(\alpha) \subseteq \overline{S}_\upsilon(\beta)$,

3) $S_\upsilon^l(\alpha) \subseteq \overline{S}_\upsilon(\beta) + S_\upsilon^l(\beta)$,

4) $S_\upsilon^r(\alpha) \subseteq \overline{S}_\upsilon(\beta) + S_\upsilon^r(\beta)$

and if $\alpha \not\gtrless \beta$, then

5) $\overline{S}_\upsilon(\alpha)\subseteq\widetilde{S}_\upsilon(\beta)$, 5′) $\overline{S}_\upsilon(\beta)\subseteq\widetilde{S}_\upsilon(\alpha)$,

6) $S_\upsilon^l(\alpha)\subseteq\widetilde{S}_\upsilon(\beta)+S_\upsilon^r(\beta)$, 6′) $S_\upsilon^l(\beta)\subseteq\widetilde{S}_\upsilon(\alpha)+S_\upsilon^r(\alpha)$,

7) $S_\upsilon^r(\alpha)\subseteq\widetilde{S}_\upsilon(\beta)+S_\upsilon^l(\beta)$, 7′) $S_\upsilon^r(\beta)\subseteq\widetilde{S}_\upsilon(\alpha)+S_\upsilon^l(\alpha)$.

5. According to Tamura [4] a β-semigroup was either

(1) a zero semigroup defined by $xy=e$ for all x, y

or (2) a semigroup which contains $w\neq e$ and which is defined by

$$wx=xw=w \quad \text{if } x\neq w;$$
$$xy=w^2=e \quad \text{if } x\neq w,\ y\neq w.$$

For convenience, we shall call (1) and (2) a β_1-semigroup and a β_2-semigroup respectively.

By the way, we can prove that $T=S_\varepsilon+S_\upsilon$ is a semigroup, which shall be called a $\bar{\beta}_1$-semigroup, defined by

$xy=x$ $[yx=x]$ for $x\in S_\varepsilon$, $y\in T$ and $x'y=e$ $[yx'=e]$ for $x'\in S_\upsilon$; $y\in T$ if S_ε is a left [right] zero.

Furthermore

Theorem 4. $G=\sum_{\nu\in\Delta_0} S_\nu$ is either

(1) a $\bar{\beta}_1$-semigroup

or (2) a β_2-semigroup.

We shall call G a $\bar{\beta}$-semigroup.

Here, we note that S_ω, S_ε, S_υ of G can be written as follows: $S_\omega=\{x\in G;\ x^2\neq x,\ x^3=x\}$, $S_\varepsilon=\{x\in G;\ x^2=x\}$, and $S_\upsilon=\{x\in G;\ x^2\neq x,\ x^3\neq x\}$, and that e, say a definite element in G, can be determined as

any fixed one element of S_ε if $S_\varepsilon=G$,

x^2, $x\in S_\omega+S_\upsilon$ if $S_\varepsilon\neq G$.

6. Combining the above theorems, we can establish the following theorem:

Theorem 5. In order that a semigroup S is a β^*-semigroup, it is necessary and sufficient that S is uniquely expressible as a partially ordered set $\Lambda=\Delta_l\cup\Delta_r\cup\Delta_0$ satisfying Theorem 2 of maximal left zero subsemigroups S_λ, $\lambda\in\Delta_l$, maximal right zero subsemigroups S_μ, $\mu\in\Delta_r$, and a non-empty maximal $\bar{\beta}$-semigroup $G=\sum_{\nu\in\Delta_0} S_\nu$ which has mutually disjoint and uniquely determined subsets $\overline{S}_\varepsilon(\alpha)$, $S_\varepsilon^l(\alpha)$ (or $S_\varepsilon^r(\alpha)$), $\overline{S}_\upsilon(\alpha)$, $S_\upsilon^l(\alpha)$, $S_\upsilon^r(\alpha)$, and $\widetilde{S}_\upsilon(\alpha)$ for all $\alpha(\not\geqq\varepsilon,\ \neq\upsilon)$ in Λ satisfying Theorem 3.

Therefore we have

Corollary 1. A β^*-semigroup S is a β-semigroup if and only if S has exactly one idempotent element.

Corollary 2. A β^*-semigroup S is a Rédei's semigroup if and only if S is a band[7] and Λ is a chain i.e. linearly ordered set.

Corollary 3. A β^*-semigroup S is a left or right zero semigroup

7) A band means a semigroup whose every element is idempotent.

if and only if S does not contain elements which commute[8] with each other at all.

7. Suppose that there are given mutually disjoint systems $\{S_\lambda\}_{\lambda\in\Delta_l}$ of mutually disjoint left zero semigroups, $\{S_\mu\}_{\mu\in\Delta_r}$ of mutually disjoint right zero semigroups and a non-empty $\bar{\beta}$-semigroup $G=\sum_{\nu\in\Delta_0} S_\nu$, $\Delta_0=\{\omega, \varepsilon, \upsilon\}$, and the suffix set $\Lambda=\Delta_l\cup\Delta_r\cup\Delta_0$ is a partially ordered set $(\geqq)$ satisfying Theorem 2 and for all $\alpha(\ngeqq\varepsilon, \neq\upsilon)$ in Λ mutually disjoint subsets $\overline{S}_\varepsilon(\alpha)$, $S_\varepsilon^l(\alpha)$, (or $S_\varepsilon^r(\alpha)$), $\overline{S}_\upsilon(\alpha)$, $S_\upsilon^l(\alpha)$, $S_\upsilon^r(\alpha)$ and $\widetilde{S}_\upsilon(\alpha)$ of G satisfying Theorem 3 are determined uniquely.

Then, put $S=\sum_{\alpha\in\Lambda} S_\alpha=\sum_{\lambda\in\Delta_l} S_\lambda+\sum_{\mu\in\Delta_r} S_\mu+\sum_{\nu\in\Delta_0} S_\nu$ and define xy, $x\in S_\alpha$ and $y\in S_\beta$, as follows:

(1) The case $\alpha=\beta$.

$xy=x\cdot y$ ($\cdot$ denotes the multiplication of G) if $\alpha\in\Delta_0$,
$=x$ if $\alpha\in\Delta_l$,
$=y$ if $\alpha\in\Delta_r$.

(2) The case $\alpha\neq\beta$ and

a) $\alpha\notin\{\varepsilon, \upsilon\}$, $\beta\notin\{\varepsilon, \upsilon\}$.

$xy=x=yx$ if $\alpha>\beta$,
$xy=y=yx$ if $\beta>\alpha$,
$xy=e=yx$ (e is the definite element of G) if $\alpha\gtrless\beta$.

b) $\alpha=\varepsilon$, $\beta\neq\upsilon$.

$xy=y=yx$ if $\beta>\varepsilon$,
$xy=x=yx$ if $\beta\ngeqq\varepsilon$ and $x\in\overline{S}_\varepsilon(\beta)$,
$xy=x$, $yx=e$ if $\beta\ngeqq\varepsilon$ and $x\in S_\varepsilon^l(\beta)$,
$xy=e$, $yx=x$ if $\beta\ngeqq\varepsilon$ and $x\in S_\varepsilon^r(\beta)$.

c) $\alpha=\upsilon$, $\beta\neq\varepsilon$.

$xy=y=yx$ if $\beta>\varepsilon$,
$xy=x=yx$ if $\beta\ngeqq\varepsilon$ and $x\in\overline{S}_\upsilon(\beta)$,
$xy=x$, $yx=e$ if $\beta\ngeqq\varepsilon$ and $x\in S_\upsilon^l(\beta)$,
$xy=e$, $yx=x$ if $\beta\ngeqq\varepsilon$ and $x\in S_\upsilon^r(\beta)$,
$xy=e=yx$ if $\beta\ngeqq\varepsilon$ and $x\in\widetilde{S}_\upsilon(\beta)$.

d) $\alpha=\upsilon$, $\beta=\varepsilon$.

$xy=x\cdot y$.

Then we can prove that S forms a β^*-semigroup with respect to the above multiplication. Thus we have

Theorem 6. Any β^*-semigroup is constructed in the above mentioned way.

8. Let $S=\sum_{\alpha\in\Lambda} S_\alpha$ and $S'=\sum_{\alpha'\in\Lambda'} S'_{\alpha'}$ be two β^*-semigroups composed by the above mentioned way. And let $G=S_\omega+S_\varepsilon+S_\upsilon$ and $G'=S'_{\omega'}+S'_{\varepsilon'}$

8) Two elements x and y are said to commute with each other if $xy=yx$.

$+S'_{\upsilon'}$ be the $\bar{\beta}$-semigroups of S and S' respectively and let $\overline{S}_\varepsilon(\alpha)$, $S^l_\varepsilon(\alpha)$, $S^r_\varepsilon(\alpha)$, $\overline{S}_\upsilon(\alpha)$, $S^l_\upsilon(\alpha)$, $S^r_\upsilon(\alpha)$, $\widetilde{S}_\upsilon(\alpha)$; $\overline{S}'_{\varepsilon'}(\alpha')$, $S'^l_{\varepsilon'}(\alpha')$, $S'^r_{\varepsilon'}(\alpha')$, $\overline{S}'_{\upsilon'}(\alpha')$, $S'^l_{\upsilon'}(\alpha')$, $S'^r_{\upsilon'}(\alpha')$, $\widetilde{S}'_{\upsilon'}(\alpha')$ be subsets of G and G' which are defined for all $\alpha(\ngeq\varepsilon, \neq\upsilon)$ in Λ and $\alpha'(\ngeq\varepsilon' \neq\upsilon')$ in Λ' respectively.

Then we have the following theorem:

Theorem 7. $S=\sum_{\alpha\in\Lambda} S_\alpha$ is isomorphic [anti-isomorphic] onto $S'=\sum_{\alpha'\in\Lambda'} S'_{\alpha'}$ if and only if there exist an order isomorphism φ of Λ onto Λ', isomorphisms [anti-isomorphisms] ψ_α of S_α onto $S'_{\varphi(\alpha)}$ for all $\alpha\in\Delta_l\cup\Delta_r$ and an isomorphism [anti-isomorphism] ψ_0 of G onto G' satisfying the following conditions: for all $\alpha(\ngeq\varepsilon, \neq\upsilon)$ in Λ

$$\psi_0(\overline{S}_\varepsilon(\alpha))=\overline{S}'_{\varepsilon'}(\varphi(\alpha)),\quad \psi_0(S^l_\varepsilon(\alpha))=S'^l_{\varepsilon'}(\varphi(\alpha))\ [\psi_0(S^l_\varepsilon(\alpha))=S'^r_{\varepsilon'}(\varphi(\alpha))],$$
$$\psi_0(S^r_\varepsilon(\alpha))=S'^r_{\varepsilon'}(\varphi(\alpha))\ [\psi_0(S^r_\varepsilon(\alpha))=S'^l_{\varepsilon'}(\varphi(\alpha))],\quad \psi_0(\overline{S}_\upsilon(\alpha))=\overline{S}'_{\upsilon'}(\varphi(\alpha)),$$
$$\psi_0(S^l_\upsilon(\alpha))=S'^l_{\upsilon'}(\varphi(\alpha))\ [\psi_0(S^l_\upsilon(\alpha))=S'^r_{\upsilon'}(\varphi(\alpha))],\quad \psi_0(S^r_\upsilon(\alpha))=S'^r_{\upsilon'}(\varphi(\alpha))$$
$$[\psi_0(S^r_\upsilon(\alpha))=S'^l_{\upsilon'}(\varphi))],\quad \psi_0(\widetilde{S}_\upsilon(\alpha))=\widetilde{S}'_{\upsilon'}(\varphi(\alpha)).$$

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