161. *On the Gibbs Phenomenon for Quasi-Hausdorff Means*

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1. The Hausdorff transformation is defined as transforming the sequence $\{s_\nu\}$ into the sequence $\{h_n\}$ by means of the equation

$$h_n=\sum_{\nu=0}^{n}\binom{n}{\nu}s_\nu\int_0^1 r^\nu(1-r)^{n-\nu}d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0\leq r\leq 1$. This transformation is regular if and only if

$$\psi(1)-\psi(0)=1,$$

and if $\psi(r)$ is continuous at $r=0$. We may assume that $\psi(0)=0$, then the above conditions become

$$\psi(1)=1,\quad \psi(+0)=\psi(0)=0.$$

Corresponding to any fixed number r with $0<r\leq 1$, if we put $\psi(x)=e_r(x)$, where

$$e_r(x)=\begin{cases}0 & \text{for} \quad 0\leq x<r\\ 1 & \text{for} \quad r\leq x\leq 1,\end{cases}$$

then the Hausdorff transformation reduces to the Euler transformation, i.e.

$$\sigma_n(r)=\sum_{\nu=0}^{n}\binom{n}{\nu}r^\nu(1-r)^{n-\nu}s_\nu.$$

The case $r=1$ corresponds to the ordinary convergence. For the fundamental properties of the Hausdorff and Euler transformations, see, e.g., G. H. Hardy ([1], Chapters VIII and XI).

Let $\phi(t)$ denote the function of period 2π and equal to $\frac{1}{2}(\pi-t)$ in the interval $0<t<2\pi$. Then $\phi(t)$ has a simple discontinuity at the origin: its Fourier series is

$$\sum_{n=1}^{\infty}\frac{\sin nt}{n}. \tag{1}$$

O. Szász [12, 13] investigated the Gibbs phenomenon of this series for the Hausdorff and Euler means. Here we put $s_0=s_0(t)=0$ and $s_\nu=s_\nu(t)=\sum_{n=1}^{\nu}\frac{\sin nt}{n}$. He proved the following

THEOREM 1. *For the regular Hausdorff means of* (1) *we have*

$$\lim_{n\to\infty}h_n(t_n)=\int_0^1 d\psi(r)\int_0^\tau \frac{\sin ry}{y}dy,$$

as $nt_n\to\tau$ *with* $0\leq\tau\leq+\infty$.

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In the following we shall also use τ as a number with $0\leq\tau\leq+\infty$. This theorem contains his earlier result on the Gibbs phenomenon for the Euler means, i.e.

THEOREM 2. *For the Euler means of* (1) *we have*

$$\lim_{n\to\infty}\sigma_n(r, t_n)=\int_0^{\tau r}\frac{\sin y}{y}dy,$$

as $nt_n\to\tau$ *and* $nt_n^2\to 0$.

2. The quasi-Hausdorff transformation has recently been investigated by B. Kuttner [4, 5, 6, 7] and M. S. Ramanujan [9, 10, 11]. This transformation is defined by means of the equation

$$h_n^*=\sum_{\nu=n}^{\infty}\binom{\nu}{n}s_\nu\int_0^1 r^{n+1}(1-r)^{\nu-n}d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0\leq r\leq 1$. This transformation is regular if and only if

$$\psi(1)-\psi(+0)=1.$$

We may assume that

$$\psi(1)=1,\quad \psi(+0)=0.$$

If we take $\psi(x)=e_r(x)$, where $e_r(x)$ is the same function as before, then the quasi-Hausdorff transformation reduces to the circle transformation, i.e.

$$\sigma_n^*(r)=\sum_{\nu=n}^{\infty}\binom{\nu}{n}r^{n+1}(1-r)^{\nu-n}s_\nu,$$

where r is any fixed number with $0<r\leq 1$. The case $r=1$ corresponds to the ordinary convergence. (See, e.g., G. H. Hardy [1], Chapters IX and XI.)

The Gibbs phenomenon of (1) for the circle means was studied by K. Ishiguro [2]. He proved the following

THEOREM 3. *For the circle means of* (1) *we have*

$$\lim_{n\to\infty}\sigma_n^*(r, t_n)=\int_0^{\tau/r}\frac{\sin y}{y}dy,$$

as $nt_n\to\tau$ *and* $nt_n^2\to 0$.

The purpose of the present note is to generalize Theorem 3 to the quasi-Hausdorff means as follows:

THEOREM 4. *For the regular quasi-Hausdorff means of* (1) *we have*

$$\lim_{n\to\infty}h_n^*(t_n)=\int_0^1 d\psi(r)\int_0^{\tau}\frac{\sin y/r}{y}dy,$$

provided that the weight function $\psi(r)$ *is continuous at* $r=0$, $nt_n\to\tau$ *and* $nt_n^2\to 0$.

3. *Proof of Theorem* 4. From the assumption we may put $\psi(1)=1$, $\psi(+0)=\psi(0)=0$. Take any fixed δ with $0<\delta<1$, and write the quasi-Hausdorff transform h_n^* as $j_n^*+k_n^*$, where

$$j_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} s_\nu \int_0^\delta r^{n+1}(1-r)^{\nu-n} d\psi(r), \tag{2}$$

and

$$k_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} s_\nu \int_\delta^1 r^{n+1}(1-r)^{\nu-n} d\psi(r). \tag{3}$$

Of course, the transformations from $\{s_\nu\}$ to $\{j_n^*\}$, $\{k_n^*\}$ are not regular, but they are "*multiplicative*" *with multipliers* $\psi(\delta)$, $1-\psi(\delta)$ respectively (i.e. they transform a sequence converging to s into sequences converging to $\psi(\delta)\cdot s$, $\{1-\psi(\delta)\}\cdot s$).

Now
$$s_0 = 0,$$
$$s_\nu = s_\nu(t) = \sum_{n=1}^{\nu} \frac{\sin nt}{n} = -\frac{t}{2} + \int_0^t \frac{\sin(\nu+\frac{1}{2})x}{2\sin\frac{1}{2}x} dx.$$

Since $\{s_\nu\}$ is bounded we get, from (3),

$$k_n^* = k_n^*(t) = \int_\delta^1 \sum_{\nu=n}^{\infty} \binom{\nu}{n} s_\nu(t) r^{n+1}(1-r)^{\nu-n} d\psi(r)$$
$$= \int_\delta^1 \sigma_n^*(r, t) d\psi(r).$$

Now, with the notation of [2], it is easily seen that the estimates

$$1-(r\rho)^{n+1} = O[(n+1)x^2],$$
$$\alpha = \frac{1-r}{r}x + O(x^3),$$

used in [2] for *fixed* r, do, in fact, hold uniformly in r for $\delta \le r \le 1$. The argument of [2] therefore shows that

$$\sigma_n^*(r, t_n) = \int_0^{nt_n/r} \frac{\sin y}{y} dy + o(1) \tag{4}$$

as $n \to \infty$, *uniformly in* r *for* $\delta \le r \le 1$. Hence we obtain easily

$$\lim_{n\to\infty} k_n^*(t_n) = \int_\delta^1 d\psi(r) \int_0^{\tau/r} \frac{\sin y}{y} dy$$
$$= \int_\delta^1 d\psi(r) \int_0^{\tau} \frac{\sin y/r}{y} dy.$$

Now it clearly follows from (2) that if $\{s_\nu\}$ is bounded, say $|s_\nu| \le M$, then

$$|j_n^*| \le M \int_0^\delta |d\psi(r)|. \tag{5}$$

In effect we have, for all ν, t

$$\left| \sum_{n=1}^{\nu} \frac{\sin nt}{n} \right| \le M, \text{ say,}$$

whence (5) holds. By the continuity of $\psi(r)$ at $r=0$ the expression (5) can be made arbitrarily small, independent of n, t, by making δ sufficiently small.

Again, we have, for all Y

$$\left|\int_0^Y \frac{\sin y}{y} dy\right| \leq M', \text{ say.}$$

Hence

$$\left|\int_0^\delta d\psi(r) \int_0^{\tau/r} \frac{\sin y}{y} dy\right| \leq M' \int_0^\delta |d\psi(r)|,$$

which again can be made arbitrarily small by making δ sufficiently small.

Collecting the above estimations we obtain

$$\lim_{n\to\infty} h_n^*(t_n) = \int_0^1 d\psi(r) \int_0^\tau \frac{\sin y/r}{y} dy,$$

whence the proof is completed.

4. *Remark.* In the previous paper [2] we proved the equation (4). But we shall make a certain simplification of its proof here. Of course, this simplification may be also applicable to the proof of Theorem 4.

We put

$$\begin{aligned}\sigma_n^*(r, t) + \frac{t}{2} &= \frac{1}{2}\int_0^t \cot \frac{1}{2}x (r\rho)^{n+1} \sin (n+1)(\alpha+x) dx \\ &\quad -\frac{1}{2}\int_0^t (r\rho)^{n+1} \cos (n+1)(\alpha+x) dx \\ &= \frac{1}{2}(I-J).\end{aligned}$$

Then we see easily $|J| \leq t$. (See [2] p. 290.)

If we write, to estimate I,

$$\alpha + x = \beta$$

we find that

$$\tan \beta = \frac{\sin x}{\cos x - (1-r)},$$

and further that

$$\beta = \frac{x}{r} + \mu x^3, \quad |\mu| \leq K \text{ say,}$$

for small x and for any fixed r with $0 < r < 1$.

The equation (4) may now be obtained as in the paper [2].

5. M. S. Ramanujan [10] introduced the transformation (S^*, ψ) by means of the equation

$$s_n^* = \sum_{\nu=0}^{\infty} \binom{n+\nu}{\nu} s_\nu \int_0^1 r^{n+1}(1-r)^\nu d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0 \leq r \leq 1$. This transformation is regular if and only if

$$\psi(1) - \psi(+0) = 1,$$

and if $\psi(r)$ is continuous at $r=1$.

If we take $\psi(x) = e_r(x)$, where $e_r(x)$ is the same function as before, then the transformation (S^*, ψ) reduces to the transformation (σ, r) or S_{1-r} of W. Meyer-König [8] and P. Vermes [14], i.e.

$$\lambda_n(r)=\sum_{\nu=0}^{\infty}\binom{n+\nu}{\nu}r^{n+1}(1-r)^{\nu}s_{\nu},$$

where r is any fixed number with $0<r<1$.

The Gibbs phenomenon of (1) for the means (σ, r) was studied by K. Ishiguro [3]. He proved the following

THEOREM 5. *For the means* (σ, r) *of* (1) *we have*

$$\lim_{n\to\infty}\lambda_n(r, t_n)=\int_0^{\frac{1-r}{r}\tau}\frac{\sin y}{y}dy,$$

as $nt_n\to\tau$ *and* $nt_n^2\to 0$.

We can generalize this theorem to the means (S^*, ψ) as follows:

THEOREM 6. *For the regular means* (S^*, ψ) *of* (1) *we have*

$$\lim_{n\to\infty}s_n^*(t_n)=\int_0^1 d\psi(r)\int_0^{\tau}\frac{\sin\frac{1-r}{r}y}{y}dy,$$

provided that the weight function $\psi(r)$ *is continuous at* $r=0$, $nt_n\to\tau$ *and* $nt_n^2\to 0$.

The proof of Theorem 6 is quite similar to that of Theorem 4.

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