7. *On the Logarithmic Property of the Indices of Endomorphism on a Linear Space*

By Perla LÓPEZ DE CICILEO and Kiyoshi ISÉKI

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Let E be a linear space, and let u be an endomorphism on E. We denote the dimension of $u^{-1}(0)$ by dim (Ker (u)), and the codimension of $u(E)$ by cod (Im(u)). If one of these values is finite, $d(u)=\dim(\mathrm{Ker}(u))-\mathrm{cod}(\mathrm{Im}(u))$ is called the *index* of u.

In his paper [1], M. Audin proved the logarithmic property of the indices:

THEOREM. *Let u, v be endomorphisms on E. If one of the indices of u, v is finite, then*

$$d(v\circ u)=d(u)+d(v).$$

In this note, we shall show that the result follows from the Deprit useful lemma [2]. We only consider that $d(u)$, $d(v)$ are finite (see M. Audin [1]).

LEMMA (*Deprit*). *Let E, F and G be linear spaces, and let $u(v)$ be a linear transformation from $E(F)$ to $F(G)$. Then E is the direct sum of three linear spaces satisfying the following conditions*:

1) $E=\mathrm{Ker}(u)\oplus E_1\oplus E_2$,
2) $\mathrm{Ker}(v\circ u)=\mathrm{Ker}(u)\oplus E_1$,
3) $\mathrm{Im}(v\circ u)=v(u(E_2))$,
4) E_1 is isomorphic with $\mathrm{Ker}(v)\cap\mathrm{Im}(u)$,
5) E_2 is isomorphic with $\mathrm{Im}(u)/\mathrm{Ker}(v)\cap\mathrm{Im}(u)$.

Let $d(u)$ and $d(v)$ be finite indices, then, from 2) and 4), we have

$$\begin{aligned}\dim(\mathrm{Ker}(v\circ u))&=\dim(\mathrm{Ker}(u))+\dim E_1\\&=\dim(\mathrm{Ker}(v))+\dim(\mathrm{Ker}(v)\cap\mathrm{Im}(u)).\end{aligned}\qquad(1)$$

On the other hand, we have an isomorphic relation:

$$F/(\mathrm{Im}(u)+\mathrm{Ker}(v))\simeq\mathrm{Im}(v)/\mathrm{Im}(v\circ u).$$

If $E=F=G$, then

$$E/(\mathrm{Im}(u)+\mathrm{Ker}(v))\simeq\mathrm{Im}(v)/\mathrm{Im}(v\circ u).$$

Hence

$$\begin{aligned}&\dim(\mathrm{Im}(v))-\dim(\mathrm{Im}(v\circ u))\\&\quad=\dim E-\dim(\mathrm{Im}(u)+\mathrm{Ker}(v))\\&\quad=\dim E-[\dim(\mathrm{Im}(u))+\dim(\mathrm{Ker}(v))-\dim(\mathrm{Im}(u)\cap\mathrm{Ker}(v))]\\&\quad=\mathrm{cod}(\mathrm{Im}(u))-\dim(\mathrm{Ker}(v))+\dim(\mathrm{Im}(u)\cap\mathrm{Ker}(v)).\end{aligned}$$

Therefore, we have

$$\begin{aligned}\dim E-\dim(\mathrm{Im}(v\circ u))=&\,\mathrm{cod}(\mathrm{Im}(u))-\dim(\mathrm{Ker}(v))\\&+\dim E-\dim(\mathrm{Im}(v))+\dim(\mathrm{Im}(u)\cap\mathrm{Ker}(v)).\end{aligned}$$

Hence

$$\operatorname{cod}(\operatorname{Im}(v\circ u))=\operatorname{cod}(\operatorname{Im}(u))-\dim(\operatorname{Ker}(v))$$
$$+\operatorname{cod}(\operatorname{Im}(v))+\dim(\operatorname{Im}(u)\cap\operatorname{Ker}(v)). \qquad (2)$$

By subtracting (2) from (1), we have the logarithmic property:

$$d(v\circ u)=d(v)+d(u).$$

References

[1] M. Audin: Sur les équations linéaires dans un espace vectoriel. Alger, Mathématique, **4**, 5–75 (1957).

[2] A. Deprit: Contribution à l'étude de l'Algèbre des applications linéaires continues d'un espace localement convexe séparé. Acad. Roy. Belgique, Cl. Sci. Mém., **31**, 1–170 (1959).