6. On Some Topologies in the Universal Hilbert Space

By Tadashige ISHIHARA
Department of Mathematics, Osaka University
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1. It is one of the most important problem of the quantum field theory that what kind of Hilbert (or linear) space we must construct corresponding to the set of all physical states, i.e., in what kind of space we must consider representation of the operator algebra of physical observables.

As many authors [1–3] pointed out, the inequivalency of irreducible representations of cannonical commutation relation (or of similar operator algebra [4–7], causes the different types of field theory. In addition to the inequivalency, the problem of orthogonality is another serious problem.

O. Miyatake [9, 10], and Van Hove [2] pointed out the problem of the orthogonality as follows. Let H_0 be the free Hamiltonian of neutral scaler meson field interacting with the fixed point source. Let H_1 be the total Hamiltonian of the coupled system. Then the Hilbert space $\boldsymbol{H}_1$ spanned by all eigenvectors of H_1 is perpendicular to the Hilbert space $\boldsymbol{H}_0$ spanned by all eigenvectors of H_0. This orthogonality deny the applicability of customary perturbation in which a state of $\boldsymbol{H}_1$ is expanded by the complete orthonormal system of $\boldsymbol{H}_0$.

Further we may well imagine that not only in the fixed source model but also in the real field similar orthogonality relation would hold, and we can imagine also well that the customary perturbative methods would break down in the real field. K. O. Friedrichs referred these problem [1] and proposed that both spaces should be considered as subspaces of a universal Hilbert space introduced by J. Von Neumann [11].

In this article we consider about universality of this space (§ 2) and the relation to the customary occupation number representation (§ 3) and investigate the new topology to obtain the new perturbative method between orthogonal spaces (§ 4–6).

2. In the quantum field theory one frequently constructs the Hilbert spaces of state vectors in the following several ways.

(1) One assumes Hamiltonian of the system, and finds the eigenvectors of Hamiltonian, then takes closure of the linear aggregate of the eigenvectors (e.g. [2, 9, 10]).

(2) One assumes the algebraic relations of operators which

corresponds to the physical observables or so. Then one assumes suitable Hilbert space which is a irreducible representation space of these algebras (e.g. [4–7]).

(3) One assumes the existence of a vector which corresponds to the vacuum state, and assumes the dense subset of Hilbert space is obtained by the iterated application of some field operators to the vacuum vector [5, 8, 12].

The construction of $\boldsymbol{H}=\Pi\otimes H_\alpha$[*)] from the primary Hilbert spaces H_α is another way. This time the construction can be done independently from field operators, contrary to above three ways, i.e., we have only to assume the existence of the spectrum of single particle and assign integer numbers to the element of spectrum. In a mathematically rigorous way J. Von Neumann shows this method, whose brief sketch is given in [13, 9, 10].

The space $\Pi\otimes H_\alpha$ has the following characters. (In this article we assume that α makes a discrete spectrum.)

(1) So far as we admit that the state of field is a composed system $\{f_\alpha\}$ of the individual state f_α, we may have to recognize that all possible states are included in this space. (The name "universal Hilbert space" by K. O. Friedrichs tells the real state of affairs.) All Hilbert spaces constructed by ways of (1), (2), (3) will be contained in the universal space. All initial Ψ and final states $S\Psi$ would naturally be included in $\boldsymbol{H}$, hence S-matrix also should be well defined operator in this space.

(2) The principle of construction of the universal space from the primary Hilbert spaces is simple and reasonable. That is to say, the inner product between $\Pi\otimes f_\alpha$ and $\Pi\otimes g_\alpha$ should be defined by $\Pi_\alpha\langle f_\alpha, g_\alpha\rangle$ which is a natural extension of the scaler product $\Pi_{\alpha=1}^{n}\langle f_\alpha, g_\alpha\rangle$ of many particle system deduced from $\langle f_\alpha, g_\alpha\rangle$ of the single particle system.

3. Let $\{\varphi_{ij}\mid j=0,1,2\cdots\}$ be a complete orthonormal system of the space H_i such that φ_{ij} corresponds to the states ψ_{nj} in [13]. Let $\Gamma_{\beta(i)}=\{\varphi_{i\beta(i)}\mid i=1,2,3,\cdots\}$, then $\Gamma_{\beta(i)}$ defines and equivalent class [11] $\widetilde{\Gamma}_{\beta(i)}$, where $\widetilde{\Gamma}_{\beta(i)}$ is a closed linear subspace spanned by $\{\varphi_{i\widetilde{\beta}(i)}\mid \widetilde{\beta}(i)=\beta(i)$ for almost all $i\}$. Let $\Pi^{\Gamma_{\beta(i)}}\otimes H_i$ be an incomplete direct product such that contains $\Pi_i\otimes H_{i\beta(i)}$. Let $H(\Gamma_{\beta(i)})$ be a Hilbert space defined in [3] and let $H(\Gamma)$ be a direct sum of $H(\Gamma_{\beta(i)})$ for all $\beta(i)$. Then we can see easily the following

Theorem 1. $H(\Gamma_{\beta(i)})=\Pi^{\Gamma_{\beta(i)}}\otimes H_i$
and $\Pi\otimes H_i \supsetneqq H(\Gamma)=\Sigma_{\beta(i)}\oplus H(\Gamma_{\beta(i)})$.

*) Notations and abbreviations in this article follow J. Von Neumann [11] and the author [13].

Proof. We can prove these properties directly from Von Neumann's investigation [11]. Inequality is proved by the following two examples.

Example 1. Let $f_i=\frac{1}{\sqrt{2}}(\varphi_{i1}+\varphi_{i2})$ for any integer i. Then $\Pi\otimes f_i$ is orthogonal to every $H(\Gamma_{\beta(i)})$, i.e. $\Pi\otimes f_i \bar{\in} H(\Gamma)$ and $\in \Pi\otimes H_i$.

Example 2. Let $f_i=\Sigma_{n=0}^{\infty}c_n\varphi_{in}$ for any i where $c_n=\sqrt{e^{-\lambda}\lambda^n/n!}$, then $\Pi\otimes f_i$ belongs to $\boldsymbol{H}_1$ (total) $\subset \boldsymbol{H}$ (universal) but neither belongs to $\boldsymbol{H}_0$ (free) nor $H(\Gamma)$.

4. Now the elimination of orthogonality is not attained, of course, by mere introduction of the universal space. We must seek the limiting process or topology τ which connect the two (free and total) subspaces of the universal space.

We desire this topology τ has the following 4 properties:

(1) Among Hilbert spaces $H(\Gamma_{\beta(i)})$, the most frequently used one is $H(\Gamma_0)^{*)}=\boldsymbol{H}_0$ (free) (Fock space). Hence we desire the closure in τ of $H(\Gamma_0)$ has as wide region as possible, since its closure is also the maximum region which can be attained by the limiting (or perturbation) process from $H(\Gamma_0)$.

(2) Topology τ must be compatible with linear (algebraic) operation between states.

(3) It is preferable to avoid troublesome modification that τ is weaker than the ordinary topology by norm.

(4) Topology τ will be preferable if the physical meaning of limiting process in τ is given or if it corresponds to the customary limiting process in the calculation of quantum field theory.

5. We investigate here the topology which corresponds to the customary cut-off process [13]. Let G be the set of all finite linear sum of c-sequence. Then G is dense subset of $\boldsymbol{H}$. We introduce the new topology in G.

In the previous article [13] we introduced the cut-off operator P_N. At first sight it seems to be reasonable to introduce a new topology using this operator P_N by a neighborhood system $U_{N,\varepsilon}(\Phi_0)$ of a vector $\Phi_0\in G$:

$$U_{N,\varepsilon}(\Phi_0)=\{\Phi \mid \|\Phi-\Phi_0\|_N\equiv\|P_N(\Phi-\Phi_0)\|<\varepsilon,\ \Phi\in G\}.$$

But this topology is not satisfactory because of the ambiguities of the cut-off process discussed in the previous paper [13].

Now there are two remedies for these ambiguities. One is to apply cut-off operation always in the standard form which is developed by a system of *fixed* complete orthonormal set. The other method is to regard a different expression as a different physical state. The similar of the former types of cut-off process is done in custom-

*) Γ_0 means $\Gamma_{\beta(i)}$ such that $\beta(i)\equiv 0$ [3].

ary calculation with no attention either to ambiguities or to remedies for them. In this article we investigate the former way. The latter way will be discussed in the following paper [14].

Now one takes a fixed base e.g. $\{\varphi_{i\beta} | i=1,2,\cdots,\beta=0,1,2,\cdots\}$ as in [13].

Let $\{\lambda_\alpha \varphi_{\alpha i_\alpha}\}$ be a c-sequence. Further assume that $\Pi\lambda_\alpha$ converges to $ke^{i\theta}$. Then we call $\Pi\otimes\lambda_\alpha\varphi_{\alpha i_\alpha}$ is normalizable and call $ke^{i\theta}\Pi\otimes\varphi_{\alpha i_\alpha}$ standard monomial of a vector $\Pi\otimes\lambda_\alpha\varphi_{\alpha i_\alpha}$. We take the set $\mathfrak{h}$ of all finite linear aggregate of normalizable vector with regard to given complete orthonormal system $\{\varphi_{i\beta}\}$. We can easily see that $\mathfrak{h}$ is an algebraic linear space. Now one can express any element Φ of $\mathfrak{h}$ by a finite linear sum of standard monomial and then one arranges it with respect to the independent basis vector $\{\Pi\otimes\varphi_{\alpha i_\alpha}\}$ after the finite steps of the following identifications:

(1) $$ke^{i\theta}\Pi\varphi_{\alpha i_\alpha}+k'e^{i\theta'}\Pi\varphi_{\alpha i_\alpha}=(ke^{i\theta}+k'e^{i\theta'})\Pi\varphi_{\alpha i_\alpha}$$

(2) $$k'e^{i\theta'}\{ke^{i\theta}\Pi\varphi_{\alpha i_\alpha}\}=kk'e^{i(\theta+\theta')}\Pi\varphi_{\alpha i_\alpha}.$$

We denote the obtained expression by $\hat{\Phi}$ and call it standard polynomial of Φ.

We introduce the topology τ in the space $\mathfrak{h}$ by the neighborhood system $U_{N,\varepsilon}(\Phi_0)=\{\Phi \mid \|\Phi-\Phi_0\|_N<\varepsilon\}$, where we denote the norm $\|P_N\hat{\Phi}\|$ of the cut-off maps $P_N\hat{\Phi}$ [13] of standardized vector $\hat{\Phi}$ of Φ by $\|\Phi\|_N$ and call it N-semi-norm, i.e, $\|\Phi\|_N=\|P_N\hat{\Phi}\|$.

Theorem 2. *$\mathfrak{h}$ is a convex linear topological space in the topology τ.*

Proof. We can easily prove this theorem from the definition of τ.

6. We compare here the topology τ with the ordinary norm in $\mathfrak{h}$ and investigate the region of $\bar{\mathfrak{h}}$.

For any monomial element Φ of $\mathfrak{h}$, we see easily that $\|\Phi\|=\|\Phi\|_N$. For polynomial element Φ such that $\hat{\Phi}=\Sigma_{\nu=1}^m\hat{\Phi}_\nu$ where $\hat{\Phi}_\nu$ is a standard monomial, we see that the inequalities $O\cdot\|\Phi\|\leqq\|\Phi\|_N\leqq\Sigma_\nu\|\Phi_\nu\|$ holds.

The 1st equality holds for $\|\Phi\|\neq 0$ as we see in Example 3 in [13]. The 2nd equality holds as we see in the following

Example 3. Let $\varphi=\varphi_1+\varphi_2$, $\varphi_1=\Psi_1\otimes\Psi_{\infty 1}$, $\varphi_2=\Psi_1\otimes\Psi_{\infty 2}$, where $\|\Psi_1\|=\|\Psi_{\infty,1}\|=\|\Psi_{\infty,2}\|=1$ (in H_1 or $\Pi_{i=2}^\infty\otimes H_i$), and $\Psi_{\infty 1}\perp\Psi_{\infty 2}$. Then we see that $\|\varphi\|_1=2=\|\varphi_1\|+\|\varphi_2\|$.

Further, as we see in Example 5 in [13], there exists an element $\Phi\in\boldsymbol{H}$ such that $\|\Phi\|<\infty$, $\|\Phi\|_N=\infty$.

With respect to the region of $\mathfrak{h}$ and $\bar{\mathfrak{h}}$, we have the following

Theorem 3. (1) *$\mathfrak{h}$ is dense in $U_\beta H(\Gamma_{\beta(i)})$,* (2) *$\bar{\mathfrak{h}}$ crosses with $H(\Gamma)$* where U_β means set-theoretical sum.

Proof. (1) is trivial from definitions of $\mathfrak{h}$ and $H(\Gamma_\beta)$. Example 5 [13] shows that there exists $\Phi=\Sigma c_\nu \Phi_\nu \in H(\Gamma_0)\subset H(\Gamma)$ and $\Phi\,\bar{\in}\,\bar{\mathfrak{h}}$. The other half of the 2nd proposition is seen from the following

Example 4. Let $\varphi^N=\varphi_N\otimes\varphi_{\infty(N)}-\varphi_N\otimes\varphi'_{\infty(N)}$ where $\varphi_{\infty(N)}\perp\varphi'_{\infty(N)}$. Then we see that though $\lim_{N\to\infty}\varphi^N=0$ in τ, φ^N diverges in $H(\Gamma)$ for suitable $\{\varphi^N\}$, i.e., $\{\varphi^N\}\,\bar{\in}\,\mathfrak{h}$ and $\bar{\in} H(\Gamma)$.

Let $\mathfrak{h}(\Gamma_{\beta(i)})=\mathfrak{h}\cap H(\Gamma_{\beta(i)})$. Then $\mathfrak{h}=U_\beta\mathfrak{h}(\Gamma_{\beta(i)})$ and $\mathfrak{h}(\Gamma_{\beta(i)})$ is dense in $H(\Gamma_{\beta(i)})$. We denote $\mathfrak{h}\cap H(\Gamma_0)$ by $\mathfrak{h}(\Gamma_0)$. Then we have the following diagram

$$\begin{array}{ccccccc} H(\Gamma_0) & \subset & U_\beta H(\Gamma_\beta) & \subset & H(\Gamma) & \subset & H \\ & & & & & \swarrow\!\!\cup & \\ \cup & & \cup & & \mathfrak{g} & & \\ & & & \swarrow\!\!\cup & & & \\ \mathfrak{h}(\Gamma_0) & \subset & \mathfrak{h} & & \subset & & \bar{\mathfrak{h}} \end{array}$$

Example 5. Let $(2,5,2,5\cdots)$ be a set of occupation number, then the sequence of vectors which correspond occupation numbers $(2,0,0,\cdots)$, $(2,5,0,0,\cdots)$, $(2,5,2,0,\cdots)$ converges in τ. Hence $\Phi(2,5,2,5\cdots)\in\bar{\mathfrak{h}}(\Gamma_0)$ though it belongs to $H(\Gamma_1)$ [3] and is orthogonal to $H(\Gamma_0)$.

Example 6. Let $\Phi_n=\Pi_{i=1}^n\otimes\frac{1}{\sqrt{2}}(\varphi_{i1}+\varphi_{i2})\otimes(\Pi_{i=n+1}^\infty\otimes\varphi_{i1})$. Then the standard polynormial $\Phi_n-\Phi_m$ for $n<m$ is

$$\left\{\left(\frac{1}{\sqrt{2}}\right)^n-\left(\frac{1}{\sqrt{2}}\right)^m\right\}\Sigma^{2^n}(\Pi_{i=1}^n\otimes\varphi_{i\lambda_i})\otimes(\Pi_{i=n+1}^\infty\otimes\varphi_{i1})$$
$$-\left(\frac{1}{\sqrt{2}}\right)^m\Sigma^{2^m-2^n}(\Pi_{i=1}^n\otimes\varphi_{i\lambda_i})\otimes(\Pi_{j=n+1}^{m-n}\otimes\varphi_{j\mu_j})\otimes(\Pi_{i=m+1}^\infty\otimes\varphi_{i1}),$$

where $\lambda_1,\cdots,\lambda_n$ runs through all the permutations of numbers 1, 2 admitting repetition, and $\mu_{n+1},\cdots,\mu_m$ is the permutation of numbers 1, 2 containing at least one number 2. Then for $N<n<m$,

$$\|\Phi_n-\Phi_m\|_N^2=(2^{n/2}-2^{m/2})^2 2^{-N}$$

and the limit of this value is indefinite. Hence $\{\Phi_n\}\in\bar{\mathfrak{h}}$. ($\{\Phi_n\}\,\bar{\in}\,H(\Gamma)$ is also to be seen in Example 1.)

Example 7. Putting $c_n=\sqrt{e^{-\lambda}\lambda^n/n!}$, we consider

$$\Phi=(\Sigma_{n=1}^\infty c_n\varphi_{1n})\otimes\Pi_{m=2}^\infty\otimes\varphi_{m1}.$$

Since c_n satisfies $\Sigma|c_n|<\infty$ we see that $\Phi\in\bar{\mathfrak{h}}$. Similarly putting $\Phi^N=\Pi_{m=1}^N(\Sigma_{n=1}^\infty c_n\varphi_{mn})\otimes\Pi_{l=N+1}^\infty\otimes\varphi_{l1}$, we see that $\Phi^N\in\bar{\mathfrak{h}}$. Φ^∞ or modified vacuum state, however, does not belong to $\bar{\mathfrak{h}}$ like Φ in Example 1.

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References

[1] K. O. Friedrichs: Mathematical Aspects of the Quantum Theory of Fields. Interscience, New York (1953).

[2] L. Van Hove: Les difficultés des divergences pour un modéle particulier de champ quantifié. Physica, **18**, 145–159 (1952).

[3] R. Haag: Canonical commutation relations in field theory and functional integration. Lectures in Theoretical Physics, Vol. III. Interscience, New York (1960).

[4] J. Von Neumann: Die Eindeutigkeit der Schrödingerschen Operatoren. Math. Ann., **104**, 570–578 (1931).

[5] H. Araki: Hamiltonian formalism and the canonical commutation relations in quantum field theory. Jour. of Math. Phy., **1**, No. 6, 492–504 (1960).

[6] I. E. Segal: Foundations of the theory of dynamical systems of infinitely many degrees of freedom. Mat. Fys. Medd. Dan. Vid. Selsk., **31** (1959), No. 12.

[7] L. Gärding and A. S. Wightman: Representations of the anticommutation relation, Representations of the commutation relation. Proc. Nat. Acad. U.S.A., **40**, 617–621, 622–626 (1956).

[8] A. S. Wightman: Quantum field theory in terms of vacuum expectation values. Phys. Rev., **101**, 860–866 (1956).

[9] O. Miyatake: On the non-existence of solution of field equations in quantum mechanics, Journal of the Institute of Polytechnics. Osaka City University, **2**, 2 (1952).

[10] ——: On the singularity of the perturbation term in the field quantum mechanics. Journal of the Institute of Polytechnics. Osaka City University, **3**, 1–2 (1952).

[11] J. Von Neumann: On infinite products. Comp. Math., **46**, 1–77 (1939).

[12] R. Haag and B. Schroer: The postulates of quantum field theory. Jour. Math. Phys., **3**, 248 (1962).

[13] T. Ishihara: On the ambiguity of cut-off process in the theory of quantum field. Proc. Japad Acad., **40**, 19–22 (1964).

[14] ——: On the universal Hilbert space in the quantum field theory. III (unpublished).