14. *On the Quasi-Hausdorff Means whose Weight Function has Jumps*

By Kazuo ISHIGURO

Department of Mathematics, Hokkaido University, Sapporo

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1. The quasi-Hausdorff transformation (H^*, ψ) is defined by means of the equation

$$h_n^* = \sum_{\nu=n}^{\infty} \binom{\nu}{n} s_\nu \int_0^1 r^{n+1}(1-r)^{\nu-n} d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0 \leq r \leq 1$. This transformation is regular if and only if

$$\psi(1) - \psi(+0) = 1.$$

We may assume that

$$\psi(1) = 1, \quad \psi(+0) = 0.$$

For the definition and the properties of this method, see, e.g., G. H. Hardy [1], B. Kuttner [3–6] and M. S. Ramanujan [9–11].

In a recent paper K. Ishiguro and B. Kuttner [2] proved the following

Theorem 1. *For the regular quasi-Hausdorff means of the Fourier series*

$$\sum_{n=1}^{\infty} \frac{\sin nt}{n}, \tag{1}$$

we have

$$\lim_{n\to\infty} h_n^*(t_n) = \int_0^1 d\psi(r) \int_0^\tau \frac{\sin y/r}{y} dy,$$

provided that the weight function $\psi(r)$ *is continuous at* $r=0$, $nt_n \to \tau$ *and* $nt_n^2 \to 0$.

This theorem corresponds to the following one of O. Szász [12].

Theorem 2. *For the regular Hausdorff means of* (1) *we have*

$$\lim_{n\to\infty} h_n(t_n) = \int_0^1 d\psi(r) \int_0^\tau \frac{\sin ry}{y} dy$$

as $nt_n \to \tau$.

Here the Hausdorff transformation (H, ψ) is defined by means of the equation

$$h_n = \sum_{\nu=0}^{n} \binom{n}{\nu} s_\nu \int_0^1 r^\nu (1-r)^{n-\nu} d\psi(r),$$

where the weight function $\psi(r)$ is of bounded variation in the interval $0 \leq r \leq 1$. This transformation is regular if, and only if,

$$\psi(1)-\psi(0)=1$$

and $\psi(r)$ is continuous at the origin. See, e.g., G. H. Hardy [1].

A. E. Livingston [7] and D. Newman [8] have recently studied the Gibbs phenomenon for this method of summability in the case where the weight function has jumps.

We shall prove, in this note, the following

Theorem 3. *If $\psi(r)$ is a step-function which is continuous at $r=0$, then the regular quasi-Hausdorff means (H^*, ψ) of* (1) *exhibit the Gibbs phenomenon.*

Proof. For the proof we use the method of D. Newman [8]. By Theorem 1, it is sufficient to prove that

$$\int_0^1 d\psi(r)\int_0^{\tau/r}\frac{\sin y}{y}\,dy>\frac{\pi}{2}$$

for some τ with $0<\tau<\infty$. Since

$$\int_0^1 d\psi(r)\int_0^{\infty}\frac{\sin y}{y}\,dy=\frac{\pi}{2},$$

it is sufficient to prove that

$$F(\tau)=\int_0^1 d\psi(r)\int_{\tau/r}^{\infty}\frac{\sin y}{y}\,dy<0$$

for some τ. This will be accomplished by showing that

(a) $\int_1^v F(\tau)d\tau$ remains bounded as $v\to\infty$,

(b) $F(\tau)\notin L^1(1,\infty)$.

We see easily, by partial integrations,

$$\int_{\tau/r}^{\infty}\frac{\sin y}{y}\,dy=\frac{r}{\tau}\cos\frac{\tau}{r}+O\left[\left(\frac{r}{\tau}\right)^2\right].$$

Hence

$$\begin{aligned}F(\tau)&=\int_0^1\left\{\frac{r}{\tau}\cos\frac{\tau}{r}+O\left[\left(\frac{r}{\tau}\right)^2\right]\right\}d\psi(r)\\&=\frac{1}{\tau}\int_0^1 r\cos\frac{\tau}{r}\,d\psi(r)+O\left(\frac{1}{\tau^2}\right),\end{aligned}\tag{2}$$

since

$$\int_0^1 r^2|d\psi(r)|\le\int_0^1|d\psi(r)|=O(1).$$

Now we shall prove (a). From (2), we have

$$\begin{aligned}\int_1^v F(\tau)d\tau&=\int_1^v\frac{d\tau}{\tau}\int_0^1 r\cos\frac{\tau}{r}\,d\psi(r)+O(1)\\&=\int_0^1 r\,d\psi(r)\int_{1/r}^{v/r}\frac{\cos\tau}{\tau}\,d\tau+O(1),\ \text{as } v\to\infty,\end{aligned}$$

where the inversion of the order of integrations is legitimate.

Here we have, for all Y,

$$\left|\int_1^Y \frac{\cos\tau}{\tau}\,d\tau\right| \leq M, \text{ say.}$$

Hence

$$\left|\int_1^v F(\tau)d\tau\right| \leq 2M\int_0^1 |d\psi(r)| + O(1) = O(1) \qquad \text{as } v\to\infty.$$

Next we shall prove (b). From the assumption, we have

$$\psi(r) = \sum_{\xi_k\leq r}\{\psi(\xi_k+0)-\psi(\xi_k-0)\},$$

where ξ_k ($\neq 0$) is the k-th discontinuity (jump) of $\psi(r)$ and the summation extends over all such (possibly countably infinite) values. From (2), we have

$$F(\tau) = \frac{1}{\tau}\sum_k \xi_k\{\psi(\xi_k+0)-\psi(\xi_k-0)\}\cos\frac{\tau}{\xi_k} + O\left(\frac{1}{\tau^2}\right)$$
$$= \frac{1}{\tau}V(\tau) + O\left(\frac{1}{\tau^2}\right), \text{ say,}$$

where $V(\tau)$ is a almost periodic function. Hence

$$\frac{1}{T}\int_0^T |V(\tau)|\,d\tau \to M > 0$$

as $T\to\infty$, since $|V(\tau)|$ is not identically equal to zero. It follows that, on integration by part,

$$\int_1^T \frac{|V(\tau)|}{\tau}\,d\tau \to M\cdot\log T$$

as $T\to\infty$, which proves (b).

2. Generally, a weight function $\psi(r)$ may be written as $\psi(r) = h(r)+j(r)$, where $h(r)$ is continuous and

$$j(r) = \sum_{\xi_k\leq r}\{\psi(\xi_k+0)-\psi(\xi_k-0)\}.$$

Then Theorem 3 may be generalized as follows:

Theorem 4. *If $\psi(r)$ is a weight function which has at least one jump and is continuous at $r=0$, and if its continuous part $h(r)$ satisfies*

$$\int_1^\infty \frac{h\left(\frac{1}{u}\right)-h(1)\frac{1}{u}}{u}\cdot\sin xu\,du \in L^1(1,\infty), \tag{3}$$

then the regular quasi-Hausdorff means (H^*,ψ) of (1) *exhibit the Gibbs phenomenon.*

Proof. Using the same argument as in the proof of Theorem 3, we see that

(a) $\displaystyle\int_1^v F(\tau)\,d\tau$ remains bounded as $v\to\infty$,

where

$$F(\tau) = \int_0^1 d\psi(r)\int_{\tau/r}^\infty \frac{\sin y}{y}\,dy$$

as before. Again, from (2),

$$F(\tau)=\frac{1}{\tau}\int_0^1 r\cos\frac{\tau}{r}\,dh(r)+\frac{1}{\tau}\int_0^1 r\cos\frac{\tau}{r}\,dj(r)+O\Big(\frac{1}{\tau^2}\Big)$$
$$=\frac{1}{\tau}\{U(\tau)+V(\tau)\}+O\Big(\frac{1}{\tau^2}\Big),$$

say. As in the proof of Theorem 3 we see

$$\int_1^T \frac{|V(\tau)|}{\tau}\,d\tau\to M\cdot\log T,\ \text{with } M>0,$$

as $T\to\infty$. Now put $h(r)=\alpha r$ with an arbitrary constant α. Then

$$U(\tau)=\alpha\int_0^1 r\cos\frac{\tau}{r}\,dr$$
$$=\alpha\tau^2\int_\tau^\infty \frac{\cos u}{u^3}\,du.$$

From the second mean value theorem of integration, we get the estimation

$$U(\tau)=O\Big(\frac{1}{\tau}\Big)$$

as $\tau\to\infty$. Hence

$$\int_1^T \frac{|U(\tau)|}{\tau}\,d\tau=O(1)$$

as $T\to\infty$.

Now we put

$$U(\tau)=\int_0^1 r\cos\frac{\tau}{r}\,dh(r)$$
$$=\int_0^1 r\cos\frac{\tau}{r}\,d\{h(r)-h(1)r\}+$$
$$+h(1)\int_0^1 r\cos\frac{\tau}{r}\,dr=U_1(\tau)+U_2(\tau),$$

where

$$\int_1^T \frac{|U_2(\tau)|}{\tau}\,d\tau=O(1).$$

Thus, to prove the theorem, it is sufficient to show

$$\int_1^T \frac{|U_1(\tau)|}{\tau}\,d\tau=O(1).$$

Here

$$U_1(\tau)=\int_0^1 r\cos\frac{\tau}{r}\,dh^*(r)$$
$$=-\int_0^1\Big(\cos\frac{\tau}{r}+\frac{\tau}{r}\sin\frac{\tau}{r}\Big)h^*(r)\,dr$$
$$=-\int_0^1\cos\frac{\tau}{r}\cdot h^*(r)\,dr-\int_0^1\frac{\tau}{r}\sin\frac{\tau}{r}\cdot h^*(r)\,dr$$
$$=-W_1(\tau)-W_2(\tau),$$

say, where $h^*(r)=h(r)-h(1)r$.

Since $h^*(r)$ is of bounded variation in the interval $0\leq r\leq 1$, it may be represented by $h_1^*(r)-h_2^*(r)$, where the functions $h_1^*(r)$ and $h_2^*(r)$ are non-decreasing and equal to zero at the origin. Further, we get

$$\int_0^1 \cos\frac{\tau}{r}\cdot h_1^*(r)\,dr=\tau\int_\tau^\infty \cos u\cdot h_1^*\left(\frac{\tau}{u}\right)\frac{du}{u^2}=O\left(\frac{1}{\tau}\right)$$

as $\tau\to\infty$. Hence

$$\int_1^T \frac{|W_1(\tau)|}{\tau}\,d\tau=O(1)$$

as $T\to\infty$.

Next

$$W_2(\tau)=\int_0^1 \frac{\tau}{r}\sin\frac{\tau}{r}\cdot h^*(r)\,dr$$

$$=\tau\int_1^\infty \frac{h^*\left(\frac{1}{u}\right)}{u}\sin\tau u\,du.$$

From the assumption (3) of the theorem, we get

$$\int_1^T \frac{|W_2(\tau)|}{\tau}\,d\tau=O(1).$$

Collecting the above estimations we see

(b) $$F(\tau)\notin L^1(1,\infty),$$

whence the proof is completed.

Remark. If we put

$$f(u)=\begin{cases}\dfrac{h\left(\frac{1}{u}\right)-h(1)\frac{1}{u}}{u} & \text{for } 1\leq u<\infty\\ 0 & \text{for } -\infty<u<1,\end{cases}$$

then the condition (3) becomes

$$\int_{-\infty}^\infty f(u)\sin xu\,du\in L^1(1,\infty).$$

By using the theorems of Fourier integrals, some further sufficient conditions on $h(r)$ may be obtained.

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