13. *Semigroups Whose Any Subsemigroup Contains a Definite Element*

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A semigroup S is called a β-semigroup if S satisfies the following two conditions:

(1) Any subset of S which contains a definite element e is a subsemigroup of S.

(2) Any subsemigroup of S contains e.

Recently T. Tamura [5] has determined all the types of β-semigroups and one of the authors [3] has done the construction of semigroups which satisfy (1).

In this paper, we shall investigate the semigroups satisfying (2). Such semigroups are called β_2^*-semigroups. A finite unipotent semigroup is a β_2^*-semigroup.

Let S be a β_2^*-semigroup and e be a definite element of S.

Lemma 1. *A subsemigroup of a β_2^*-semigroup is a β_2^*-semigroup.*

Lemma 2. *A homomorphic image of a β_2^*-semigroup is a β_2^*-semigroup.*

Lemma 3. *S is a unipotent inversible* [4].

Proof. Since $\langle e^2\rangle$ is a subsemigroup of S, it follows that $e\in\langle e^2\rangle$ because of (2), hence $\langle e\rangle$ is a finite cyclic semigroup and contains an idempotent f, and hence $e=f$ since $\langle f\rangle=\{f\}\ni e$. And for any a of S, since $e\in\langle a\rangle$, there exists a positive integer n such that $a^n=e$. Thus, we get this lemma.

Accordingly, by the theory of [4] we have

Lemma 4. *S contains a greatest periodic group G $(=eS=Se)$ as a least ideal.*

Lemma 5. *The difference semigroup $(S:G)$ of S modulo G, in Rees' sense* [2], *is a nilpotent, where by a nilpotent we mean a semigroup with unique idempotent which is a zero* 0 *and satisfies that for any element a there exists a positive integer n such that $a^n=0$.*

Thus, we have

Theorem 1. *A semigroup S is a β_2^*-semigroup if and only if S contains a periodic subgroup G such that $(S:G)$ is a nilpotent.*

Proof. We shall prove the sufficiency only. Let T be any subsemigroup of S. Then we get easily $T\cap G\neq\Box$. Hence we can take $x\in T\cap G$ and $\langle x\rangle\subseteq T\cap G$.

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And it follows that $\langle x\rangle \ni e$ (the identity of G) since G is a periodic, hence $e\in T$.

We can prove that G is a unique if it exists.

Using [4], we have immediately

Theorem 2. *Given a periodic group G and a nilpotent semigroup Z which is disjoint from G and given a ramified homomorphism φ of the set Z^* of all non-zero elements of Z into G, we can construct uniquely a β_2^*-semigroup S having G as its greatest group and its least ideal and $(S:G)$ is isomorphic to Z.*

The above S is denoted by (G, Z, φ).

The isomorphism problem on (G, Z, φ) is also solved by the same way in [4].

Theorem 3. *A β_2^*-semigroup is a β-semigroup if and only if*

(i) *the order of G is at most 2,*

(ii) *Z is a zero semigroup*

and (iii) *$Z^*\varphi=e$ (the identity of G).*

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