40. On Bückner's Inclusion Theorems for Hermitean Operators

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1. Let

$$A=\int_{-\infty}^{+\infty} t\, d\, E_t \tag{1}$$

be an Hermitean operator on a Hilbert space H. For a vector u with the norm unity, the so-called *Schwarz constants* is defined by

$$a_n=(A^n u, u), \qquad n=0, 1, 2, \cdots. \tag{2}$$

It is obvious that a_n is the n-th moment of the distribution function

$$m(t)=(E_t u, u). \tag{3}$$

According to the spectral theorem, the measure dm defined by $m(t)$ is condensed on the spectrum of A.

In numerical analysis, it is sometimes important to know that a spectre of A is contained in an interval whose end points are determined by functions of the Schwarz constants. Some theorems of such a kind which are called the *inclusion theorems* are systematically obtained by Bückner, Wielandt and the others (cf. [2; § 12.5] where detailed references are included) for a completely continuous Hermitean operators. In the present note, these inclusion theorems will be generalized for an Hermitean operator with a few modifications. It may be interested that the inclusion theorems contains the well-known Krylov-Weinstein's theorem (cf. [1] and [6]) as a special case.

2. The following theorem is fundamental for Bückner's inclusion theorems:

THEOREM 1. *Each of the sets* $\{t; t\leqq a_1\}$ *and* $\{t; t\geqq a_1\}$ *contains at least one spectre of* A. *If moreover* u *is not a proper vector belonging to* a_1, *then each of the sets* $\{t; t<a_1\}$ *and* $\{t; t>a_1\}$ *contains at least a spectre of* A.

The proof of the theorem requires a minor modification of that of Bückner [2; Thm. 12.1]. If u is a proper vector belonging to a_1, then the theorem is obvious. Now, suppose that u is not a proper vector belonging to a_1. Then the measure dm cannot concentrate at a_1. Consequently, if the spectrum of A is contained in $\{t; t\leqq a_1\}$, then

$$a_1=(Au, u)=\int_{-\infty}^{\infty} t\, d(E_t u, u)=\int_{-\infty}^{a_1} t\, dm<a_1.$$

This contradiction proves the theorem.

A polynomial

$$p(t)=p_0+p_1t+\cdots+p_{n-1}t^{n-1} \tag{4}$$

with the real coefficients will be called an *inclusion polynomial* in the sense of Bückner and Wielandt if

$$p_0a_0+p_1a_1+\cdots+p_{n-1}a_{n-1}=0. \tag{5}$$

It is easy to see that a real polynomial $p(t)$ is an inclusion polynomial if and only if

$$(p(A)u, u)=0. \tag{6}$$

If $p(A)u\neq 0$, then $p(t)$ is called *proper.*

THEOREM 2. *If $p(t)$ is an inclusion polynomial with real roots, then each of the sets $\{t; p(t)\leqq 0\}$ and $\{t; p(t)\geqq 0\}$ contains a spectre of A. Moreover, if $p(t)$ is proper and if the sets $\{t; p(t)<0\}$ and $\{t; p(t)>0\}$ are non-void, then each of the sets contains a spectre of A.*

The following proof is a reproduction of that of Bückner [2; Thm. 12.2]. Put $q(t)=p(t)-p_0$ and $B=q(A)$. (6) implies

$$(Bu, u)=((p(A)-p_0)u, u)=-p_0.$$

Since $p(t)$ is proper, $Bu=p(A)u-p_0u\neq -p_0u$, whence u is not a proper vector of B belonging to $-p_0$. Therefore Theorem 1 implies that each of the sets $\{t; t-p_0<0\}$ and $\{t; t-p_0>0\}$ contains a spectre of B, say t_1 and t_2 respectively. By a theorem on the spectrum of Hermitean operators (e.g. [3]), there are spectres s_1 and s_2 of A such that $q(s_1)=t_1$ and $q(s_2)=t_2$. Hence $p(s_1)=q(s_1)-p_0<0$ and $p(s_2)=q(s_2)-p_0>0$, as desired.

THEOREM 3 ([2, Thm. 12.3]). *If k is even, if A^ku, $A^{k+1}u$ and $A^{k+2}u$ are linearly independent, and if $a_{k+1}-b\,a_{k+2}\neq 0$ for a real number b, then a spectre of A lies between $1/b$ and $1/c$, where*

$$c=\frac{a_k-b\,a_{k+1}}{a_{k+1}-b\,a_{k+2}}.$$

Let $p(t)=p_kt^k+p_{k+1}t^{k+1}+p_{k+2}t^{k+2}$ where the coefficients satisfy

$$a_kp_k+a_{k+1}p_{k+1}+a_{k+2}p_{k+2}=0$$

and

$$p_k+p_{k+1}\frac{1}{b}+p_{k+2}\frac{1}{b^2}=0.$$

It is not hard to see that $p(t)$ is an inclusion polynomial and that $1/b$ and $1/c$ are the roots of $p(t)=0$. Since k is even, Theorem 2 shows that there exists a spectre t' of A satisfying $p(t')<0$, whence t' lies between $1/b$ and $1/c$.

THEOREM 4. *The interval $[a_1-s, a_1+s]$ contains at least one spectre of A, where*

$$s=a_2-a_1^2=||A\,u||^2-(Au, u)^2.$$

If u, Au, and A^2u are linearly independent, then the open interval

(a_1-s, a_1+s) *contains a spectre.*

This is the well-known theorem of Krylov-Weinstein which follows from Theorem 2: Put

$$p(t)=(2a_1^2-a_2)-2a_1t+t^2.$$

Clearly, $p(t)$ is an inclusion polynomial whose roots are a_1-s and a_1+s, whence Theorem 2 implies Theorem 4.

3. In the remainder, an application of the theory of orthogonal polynomials on the inclusion theorems will be briefly discussed. For the sake of simplicity, let us assume that $u, Au, A^2u, \cdots$ *are linearly independent and span* H. Consequently, every inclusion polynomial is proper and the spectrum of A coincides with the support of the measure dm.

THEOREM 5. *If* $p_0(t), p_1(t), p_2(t), \cdots$ *are the orthogonal polynomials associated with* dm, *and if*

$$t_1<t_2<\cdots<t_n$$

are the roots of $p_n(t)=0$, *then each open interval* (t_i, t_{i+1}) *contains a spectre of* A.

This is a consequence of the well-known separation theorem [4; p. 50] which states that $m(t)$ cannot be a constant in (t_i, t_{i+1}).

Theorem 5 is also obtained by Bückner and the others (cf. [2; p. 453] and also [5]). Bückner proved the theorem using his inclusion theorems since the orthogonal polynomials [4; p. 27]

$$p_n(t)=\begin{vmatrix} 1 & t & \cdots & t^n \\ a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1} & a_n & \cdots & a_{2n-1} \end{vmatrix}$$

are inclusion polynomials in the sense of the above.

THEOREM 6. *If* b *and* c *are the spectres of a bounded Hermitean operator* A *such that* $\|E_b u\|<\|E_c u\|$, *then there exists a root* d *of* $p_n(t)=0$ *for sufficiently large* n *satisfying* $b\leqq d\leqq c$.

By a theorem on the orthogonal polynomials [4; p. 110], there is a root d of $p_n(t)=0$ in $[b, c]$ for sufficiently large n if dm distributes on a finite interval and $dm(t)=\|E_c u\|^2-\|E_d u\|^2>0$, whence the theorem follows.

References

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