31. A Local Asymptotic Law for the Transient Stable Process

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In a preceding paper [5], the author has obtained a criterion of an upper class and a lower class concerning the asymptotic behaviours of the transient symmetric stable process when the time parameter t tends to infinity. The concept of the upper class and the lower class with respect to certain stochastic processes can be defined also for a neighbourhood of $t=0$ as for the case of $t=\infty$. In fact, S. Watanabe and the author, in [6], have given two criteria with respect to Cauchy process on a line. For the Brownian motion it is well-known that, by virtue of the principle of projective invariance of P. Lévy [4], the criterion for the former can be derived from that for the latter and *vice versa*. Unfortunately such a principle is unknown for stable process and it does not seem that the generalization of Lévy proof for such a process is possible. So in this paper we shall give the criterion for local case of the transient stable process directly.

There are various generalization of the "infinite-sum" part of the Borel-Cantelli lemmas for dependent events. Among them, that of Chung-Erdös [1] is the most celebrated. Recently J. Lamperti [3] obtained a very simple lemma and essentially the same result is also indicated by Ciesielski-Taylor [2]. However, their lemmas are not delicate enough to deal with our problem. Suggested by their results, we show in Lemma B the following fact that the condition (ii) of Chung-Erdös [1] can be removed if we assume that 0–1 law is valid.

Let $\{X(t, w);\ 0\leqq t<\infty\}$ be the symmetric stable process of index α in R^N. We concern only with the transient case, that is, the case $\alpha<N$. As usual we assume $X(0, w)=0$ with probability one. For any positive monotone non-increasing function $g(t)$ defined for large t's, we put

$$F(w)=\left\{t;\ |X(t, w)|\leqq t^{1/\alpha}g\left(\frac{1}{t}\right)\right\}$$

and if

$$P\{w;\ \inf_{t\in F(w)} t>0\}=0 \qquad (\text{or } =1) \tag{1}$$

then we say that $g(t)$ belongs to the upper class $\mathfrak{U}_0$ (or the lower class $\mathfrak{L}_0$) with respect to $X(t)$. Then we show the following

Theorem. *A positive monotone non-increasing function $g(t)$ belongs to the lower class $\mathfrak{L}_0$ or to the upper class $\mathfrak{U}_0$ according as the integral*

$$\int^{\infty}\frac{1}{t}\{g(t)\}^{N-\alpha}dt \tag{2}$$

is convergent or divergent.

Corollary. *For any $\delta>0$, we have*

$$P\left\{\lim_{t\to 0}\frac{|X(t,w)|}{t^{1/\alpha}|\log t|^{-\frac{1+\delta}{N-\alpha}}}=\infty\right\}=1$$

and

$$P\left\{\liminf_{t\to 0}\frac{|X(t,w)|}{t^{1/\alpha}|\log t|^{-\frac{1}{N-\alpha}}}=0\right\}=1.$$

First of all we state, for convenience of reference, two lemmas which we need in the sequel.

Lemma A. *For any $r>0$, $T\geqq 0$ and $K>1$, put*

$$H(T,KT;r)=P\{|X(t)|\leqq r \quad \text{for some} \quad T<t\leqq KT\}.$$

Then we have

$$H(T,KT;r)\leqq c_1\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}, \tag{3}$$

where c_1 is a positive constant.

Further there exist a positive integer c and a positive constant c_2 such that

$$H(T,KT;r)\geqq c_2\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha} \tag{4}$$

holds for every $T^{1/\alpha}\geqq r$ and *$K\geqq c>1$.*

Proof. Applying Lemma 4 of [5], we obtain

$$H(T,KT;r)\leqq P\{|X(t)|\leqq r \quad \text{for some} \quad t>T\}\leqq c_1\left(\frac{r}{T^{1/\alpha}}\right)^{N-\alpha}.$$

The inequality (4) is just Lemma 5 of [5]. (The c's in the present paper are not, of course, the same as in [5].)

Lemma B. *Let $\{E_k\}$ be a sequence of events satisfying* (i), (ii) *and* (iii):

(i) $\sum_{k=1}^{\infty}P(E_k)=\infty$.

(ii) $P(\limsup E_k)=0$ *or* 1.

(iii) *There exist two absolute constants c_3 and c_4 with the following property: to each E_j there corresponds a set of events $E_{j_1},\cdots,E_{j_s}$ belonging to $\{E_k\}$ such that*

$$\sum_{i=1}^{s}P(E_j\cap E_{j_i})<c_3P(E_j) \tag{5}$$

and that for any other E_k than E_{j_i} $(1\leqq i\leqq s)$ which stands after E in the sequence (viz. $k>j$), the inequality

$$P(E_j \cap E_k) < c_4 P(E_j)P(E_k) \tag{6}$$

holds. Under these conditions, infinitely many events E_k occur with probability 1.

Proof. Because of (ii), it is sufficient to prove that $P(\limsup E_k) > 0$, that is, $P(\bigcup_{k=h}^{\infty} E_k) > 0$ as $h \to \infty$. Suppose that this is not true. Then for any $\varepsilon > 0$, there exists some h such that $P(\bigcup_{k=h}^{\infty} E_k) < \varepsilon$. On the contrary, $\sum_{k=h}^{\infty} P(E_k) = \infty$ by virture of (i). Hence we can find an integer $n > h$ such that

$$1 < \sum_{k=h}^{n} P(E_k) \leqq 2. \tag{7}$$

Using (iii), we obtain

$$\sum_{h \leqq j < k \leqq n} P(E_j \cap E_k) \leqq c_3 \sum_{j=h}^{n} P(E_k) + c_4 \sum_{h \leqq j < k \leqq n} P(E_j)P(E_k)$$
$$\leqq 2c_3 + 1/2 \cdot c_4 \cdot 2^2 = c_5. \tag{8}$$

Chung-Erdös [1] proved the following inequality: Let $\{F_k\}$, $k = 1, \cdots, n$, be an arbitrary sequence of events. Then

$$2P(\bigcup_{k=1}^{n} F_k) \cdot \sum_{1 \leqq j < k \leqq n} P(F_j \cap F_k) \geqq (\sum_{k=1}^{n} P(F_k))^2 - P(\bigcup_{k=1}^{n} F_k) \cdot \sum_{k=1}^{n} P(F_k)$$

holds. Applying this to our $\{E_k\}$, $h \leqq k \leqq n$, we have by (7) and (8)

$$2\varepsilon c_5 \geqq 1 - 2\varepsilon.$$

Since ε may be taken arbitrarily small, this inequality is absurd. This completes the proof.

We are now ready to prove the Theorem. The method of the proof is somewhat routine and similar arguments are discussed in detail in [5]; so we sketch it here.

For any positive integers k, we set

$$\overline{E}_k = \{w;\ |X(t)| \leqq c^{\frac{-k+1}{\alpha}} g(c^{-k+1}) \text{ for some } c^{-k} < t \leqq c^{-k+1}\}$$
$$E_k = \left\{w;\ |X(t)| \leqq t^{1/\alpha} g\left(\frac{1}{t}\right) \text{ for some } c^{-k} < t \leqq c^{-k+1}\right\}$$

and

$$\underline{E}_k = \{w;\ |X(t)| \leqq c^{-k/\alpha} g(c^{-k}) \text{ for some } c^{-k} < t \leqq c^{-k+1}\}.$$

Then it follows from Lemma A

$$P(\overline{E}_k) \leqq c_1 \{g(c^k)\}^{N-\alpha} \tag{9}$$

and

$$P(\underline{E}_k) \geqq c_2 \{c^{-1/\alpha} g(c^k)\}^{N-\alpha} \tag{10}$$

provided that $g(c^k) \leqq 1$.

Assume that the integral in (2) converges. Then we can show $g(t) \in \mathfrak{L}_0$ by means of (9) and the Borel-Cantelli Lemma.

In the same way, when the integral diverges, it is seen by (10) that

$$\sum_{k=1}^{\infty} P(E_k) \geqq \sum_{k=1}^{\infty} P(\underline{E}_k) \geqq O(1) \int_1^{\infty} \frac{1}{t} \left\{g(t)\right\}^{N-\alpha} dt = \infty.$$

The probability in (1) is always 0 or 1 by Blumenthal's 0–1 law. Let $j<k$. When $k-j$ is sufficiently large, we can justify the inequality (6) as in [5]. In the present case the rôles of j and k should be reversed as compared with the case of $t=\infty$. To prove the inequality (5), it suffices to observe that

$$P(\underline{E}_j \cap \underline{E}_k) \leqq P(\underline{E}_j)$$

and that the k's such that $k-j$ is not large form a finite set.

Remark. Let $B(t)$ be the Wiener process. It is well-known that $B(t)/\sqrt{t}$ and $\sqrt{t}\,B(1/t)$ $(0<t<\infty)$ have the same joint distribution as a special case of the projective invariance. The criteria for the neighbourhoods of ∞ and 0, which we have obtained in [5], [6] and this paper, suggest to us that $X(t)/t^{1/\alpha}$ and $t^{1/\alpha}X(1/t)$ $(0<t<\infty)$ have the same joint distribution for stable process.

References

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