87. On Extension of Linear Functional on Abelian Group

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In this note, we shall prove an extension theorem of linear functional on an Abelian group with some restriction.

LEMMA. *Let $p(x)$, $q(x)$ be real-valued functionals on an Abelian group G satisfying the following conditions*:

(1) $p(x+y)\leq p(x)+p(y)$, and $p(0)=0$,

(2) $q(x+y)\geq q(x)+q(y)$, and $q(0)=0$.

Let H be a subgroup of G and let $f(x)$ be a linear functional on H, i.e. $f(x+y)=f(x)+f(y)$. *Suppose that*

(3) $q(x)\leq f(x)\leq p(x)$

on H. If for an element $a\in G-H$, there is a positive integer m such that $ma\in H$, $f(x)$ has a linear extension $F(x)$ on H' satisfying $q(x)\leq F(x)\leq p(x)$, where H' is the subgroup generated by H and the element a.

To prove Lemma, let k be the least positive integer m such that $ma\in H$. Then $k\geq 2$ and $nka\in H$ for $n=1,2,\cdots$. Hence by (1), (2), and (3), we have

$$kq(x\pm na)\leq f(kx\pm nka)=kf(x)\pm nf(ka)$$

and

$$kf(x)\pm mf(ka)=f(kx\pm mka)\leq kp(x\pm ma)$$

for $m, n=1,2,\cdots$ and $x\in H$. Therefore for all $x\in H$, we have

$$\frac{f(x)-p(x-ma)}{m}, \frac{q(x+na)-f(x)}{n}\leq\frac{f(ka)}{k}=f(a)$$
$$\leq\frac{p(x+ma)-f(x)}{m}, \frac{f(x)-q(x-na)}{n}.$$

Define $F(a)=f(a)$, then $F(x\pm ma)=f(x)\pm mF(a)$ $(m=0,1,2,\cdots)$ satisfies $q(x\pm ma)\leq F(x\pm ma)\leq p(x\pm ma)$ by the inequality above. Therefore we complete the proof of Lemma.

An immediate result of Lemma is

Theorem. *Assume that p, q, f and G, H satisfy the condition of Lemma. If G/H is a torsion group, then the linear functional f has a linear extension $F(x)$ on G such that $q(x)\leq F(x)\leq p(x)$ on G.*