85. *Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XI*

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In this paper we are again concerned with the problem of applying Theorem 3 [cf. Proc. Japan Acad., Vol. 38, No. 6, 267–268 (1962)] from a different point of view.

Theorem 28. Let M be a positive constant; let $\mathfrak{H}$, (β_{ij}), and $\mathfrak{F}(M)$ be the same notations as those defined in the preceding paper; let $\{\lambda_\nu\}_{\nu=1,2,3,\cdots}$ be an arbitrarily prescribed infinite sequence of complex numbers (counted according to the respective multiplicities) such that $\sup_\nu|\lambda_\nu|\leqq M$; let $\{\varphi_\nu\}_{\nu=1,2,3,\cdots}$ and $\{\psi_\mu\}_{\mu=1,2,3,\cdots}$ both be incomplete orthonormal systems which are mutually orthogonal and determine a complete orthonormal system in $\mathfrak{H}$; let c be an arbitrarily given complex number, not zero; let N be the bounded normal operator defined by

$$N=\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}+c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}\quad(\Psi_\mu=\sum_{j=1}^{\infty}\beta_{\mu j}\psi_j);$$

let Γ be a rectifiable closed Jordan curve, positively oriented, such that the disk $|\lambda|\leqq\max[M,|c|\cdot\|(\beta_{ij})\|]$ lies in the interior of Γ itself; and let $(\beta_{ij})^n=(\beta_{ij}^{(n)})$, $(n=0,1,2,\cdots;\ \beta_{ij}^{(0)}=0$ for $i\neq j$; $\beta_{jj}^{(0)}=1$ for $j=1,2,3,\cdots;\ \beta_{ij}^{(1)}=\beta_{ij})$, for convenience. Then, for the ordinary part $R_\omega(\lambda)=\sum_{n\geqq0}a_\omega^{(n)}\lambda^n$, $(|\lambda|<\infty)$, of any $S_\omega(\lambda)\in\mathfrak{F}(M)$,

$$(26)\quad \begin{aligned}\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)(\lambda I-N)^{-1}d\lambda&=\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)\varphi_\nu\otimes L_{\varphi_\nu}+\sum_{\mu=1}^{\infty}\sum_{n\geqq0}a_\omega^{(n)}c^n\Psi_\mu^{[n]}\otimes L_{\psi_\mu}\\&\equiv T\quad(i=\sqrt{-1}),\end{aligned}$$

where $\Psi_\mu^{[n]}=\sum_{j=1}^{\infty}\beta_{\mu j}^{(n)}\psi_j$ and the linear functional-series T on the right-hand side is a bounded normal operator with point spectrum $\{R_\omega(\lambda_\nu)\}_{\nu=1,2,3,\cdots}$ in $\mathfrak{H}$. Moreover the eigenspace of T corresponding to the eigenvalue $R_\omega(\lambda_\nu)$ coincides with that of N corresponding to the eigenvalue λ_ν.

Proof. Since, by hypotheses, (β_{ij}) is a bounded normal matrix-operator with $\sum_{j=1}^{\infty}|\beta_{ij}|^2\neq|\beta_{ii}|^2>0$, $(i=1,2,3,\cdots)$, in Hilbert coordinate space l_2, the point spectrum of N is surely given by $\{\lambda_\nu\}$, as already demonstrated before [cf. Proc. Japan Acad., Vol. 39, No. 10, 743–748 (1963)]. Moreover, by hypotheses, all the singularities of $S_\omega(\lambda)$ and the (point and continuous) spectra of N are wholly contained in the interior of Γ. By reference to Theorem 3, we have therefore

$$(27)\qquad \frac{1}{2\pi i}\int_{\Gamma} S_{\omega}(\lambda)(\lambda I-N)^{-1}d\lambda = R_{\omega}(N) \quad (i=\sqrt{-1}).$$

Since, on the other hand, the operators $\sum_{\nu=1}^{\infty}\lambda_\nu\varphi_\nu\otimes L_{\varphi_\nu}$ and $c\sum_{\mu=1}^{\infty}\Psi_\mu\otimes L_{\psi_\mu}$ are orthogonal to each other, the following relations hold for every $f\in\mathfrak{H}$:

$$\begin{aligned}
N^2 f &= N\Big(\sum_{\nu=1}^{\infty}\lambda_\nu(f,\varphi_\nu)\varphi_\nu + c\sum_{\mu=1}^{\infty}(f,\psi_\mu)\Psi_\mu\Big)\\
&= \sum_{\nu=1}^{\infty}\lambda_\nu\Big(\sum_{\kappa=1}^{\infty}\lambda_\kappa(f,\varphi_\kappa)\varphi_\kappa,\varphi_\nu\Big)\varphi_\nu\\
&\quad + c\sum_{\mu=1}^{\infty}\Big(c\sum_{\kappa=1}^{\infty}(f,\psi_\kappa)\Psi_\kappa,\psi_\mu\Big)\Psi_\mu\\
&= \sum_{\nu=1}^{\infty}\lambda_\nu^2(f,\varphi_\nu)\varphi_\nu + c^2\sum_{\mu=1}^{\infty}\Big[\sum_{j=1}^{\infty}(f,\psi_j)\beta_{j\mu}\Big]\Psi_\mu\\
&= \sum_{\nu=1}^{\infty}\lambda_\nu^2(f,\varphi_\nu)\varphi_\nu + c^2\sum_{j=1}^{\infty}(f,\psi_j)\Psi_j^{[2]},
\end{aligned}$$

where $\Psi_j^{[2]}=\sum_{\mu=1}^{\infty}\beta_{j\mu}\Psi_\mu$. If we now suppose that the relation

$$N^p f = \sum_{\nu=1}^{\infty}\lambda_\nu^p(f,\varphi_\nu)\varphi_\nu + c^p\sum_{\mu=1}^{\infty}(f,\psi_\mu)\Psi_\mu^{[p]}$$

holds for some positive integer p, then, in the same manner as above, we have

$$\begin{aligned}
N^{p+1}f &= N^p\Big[\sum_{\nu=1}^{\infty}\lambda_\nu(f,\varphi_\nu)\varphi_\nu + c\sum_{\mu=1}^{\infty}(f,\psi_\mu)\Psi_\mu\Big]\\
&= \sum_{\nu=1}^{\infty}\lambda_\nu^{p+1}(f,\varphi_\nu)\varphi_\nu + c^{p+1}\sum_{j=1}^{\infty}(f,\psi_j)\Psi_j^{[p+1]},
\end{aligned}$$

where $\Psi_j^{[p+1]}=\sum_{\mu=1}^{\infty}\beta_{j\mu}\Psi_\mu^{[p]}$. Consequently it is found by mathematical induction that the relation

$$N^n f = \sum_{\nu=1}^{\infty}\lambda_\nu^n(f,\varphi_\nu)\varphi_\nu + c^n\sum_{\mu=1}^{\infty}(f,\psi_\mu)\Psi_\mu^{[n]}$$

holds for any non-negative integer n, on the assumption that $\Psi_\mu^{[1]}$ and $\Psi_\mu^{[0]}$ denote Ψ_μ and ψ_μ respectively. Since this result is valid for every $f\in\mathfrak{H}$,

$$(28)\qquad N^n = \sum_{\nu=1}^{\infty}\lambda_\nu^n\varphi_\nu\otimes L_{\varphi_\nu} + c^n\sum_{\mu=1}^{\infty}\Psi_\mu^{[n]}\otimes L_{\psi_\mu}.$$

In addition to it, we can show as below that the relation $\Psi_\mu^{[n]}=\sum_{j=1}^{\infty}\beta_{\mu j}^{(n)}\psi_j$ holds for any non-negative integer n by virtue of the hypothesis concerning $(\beta_{ij}^{(n)})$. In fact, it is easily verified by direct computation that

$$\begin{aligned}
\Psi_\mu^{[2]} &= \sum_{j=1}^{\infty}\beta_{\mu j}\Psi_j\\
&= \sum_{j=1}^{\infty}(\beta_{\mu 1}\beta_{1j}+\beta_{\mu 2}\beta_{2j}+\beta_{\mu 3}\beta_{3j}+\cdots)\psi_j\\
&= \sum_{j=1}^{\infty}\beta_{\mu j}^{(2)}\psi_j
\end{aligned}$$

and that if $\Psi_\mu^{[p]}=\sum_{j=1}^{\infty}\beta_{\mu j}^{(p)}\psi_j$ for some positive integer p,

$$\Psi_\mu^{[p+1]}=\sum_{j=1}^{\infty}\beta_{\mu j}\Psi_j^{[p]}$$
$$=\sum_{j=1}^{\infty}(\beta_{\mu 1}\,\beta_{1j}^{(p)}+\beta_{\mu 2}\,\beta_{2j}^{(p)}+\beta_{\mu 3}\,\beta_{3j}^{(p)}+\cdots)\psi_j$$
$$=\sum_{j=1}^{\infty}\beta_{\mu j}^{(p+1)}\psi_j.$$

Hence, by mathematical induction, the relation $\Psi_\mu^{[n]}=\sum_{j=1}^{\infty}\beta_{\mu j}^{(n)}\psi_j$ holds for any non-negative integer n. Since, by the definition concerning the ordinary part of $S_\omega(\lambda)$, $R_\omega(\lambda)$ is expressible in the form $R_\omega(\lambda)=\sum_{n\geqq 0}a_\omega^{(n)}\lambda^n$ on the domain $\{\lambda: |\lambda|<\infty\}$, the applications of (28) and the just established result to (27) yield the required relation (26).

The proof of the latter half of the theorem is trivial. If we now denote by $\{K(\lambda)\}$ the complex spectral family of N, then

$$T=R_\omega(N)=\int_{|\lambda|\leqq\|N\|}R_\omega(\lambda)dK(\lambda)$$

and

$$T^*=\int_{|\lambda|\leqq\|N\|}\overline{R_\omega(\lambda)}dK(\lambda).$$

Hence

$$TT^*=T^*T=\int_{|\lambda|\leqq\|N\|}|R_\omega(\lambda)|^2dK(\lambda);$$

and $\|T\|\leqq\max_{|\lambda|=\|N\|}|R_\omega(\lambda)|$, that is, $\|T\|$ never exceeds the maximum modulus of $R_\omega(\lambda)$ on the circumference $|\lambda|=\max[\sup_\nu|\lambda_\nu|, |c|\cdot\|(\beta_{ij})\|]$ because of the relation $\|N\|=\max[\sup_\nu|\lambda_\nu|, |c|\cdot\|(\beta_{ij})\|]$ shown before. It also is a direct consequence of the above spectral expression of T that the point spectrum of T is given by $\{R_\omega(\lambda_\nu)\}_{\nu=1,2,3,\dots}$ and that the eigenspace of T corresponding to the eigenvalue $R_\omega(\lambda_\nu)$ coincides with that of N corresponding to the eigenvalue λ_ν.

With these results the proof of the theorem is complete.

Corollary 5. Let $L_2(\Delta,\mu)$ be the same notation as that defined in Corollary 4; let M, $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$, $\mathfrak{F}(M)$, c, (β_{ij}), and $(\beta_{ij}^{(n)})$ be also the same notations as those used in Theorem 28; let $\{\varphi_\nu(x)\}_{\nu=1,2,3,\dots}$ and $\{\psi_\kappa(x)\}_{\kappa=1,2,3,\dots}$ both be incomplete orthonormal systems which are orthogonal to each other and determine a complete orthonormal system in $L_2(\Delta,\mu)$; let N be the bounded normal operator defined by

$$(Nf)(x)=\sum_{\nu=1}^{\infty}\lambda_\nu\int_\Delta f(y)\overline{\varphi_\nu(y)}d\mu(y)\cdot\varphi_\nu(x)+c\sum_{\kappa=1}^{\infty}\int_\Delta f(y)\overline{\Psi_\kappa(y)}\,d\mu(y)\cdot\Psi_\kappa(x),$$

where $\Psi_\kappa(x)=\sum_{j=1}^{\infty}\beta_{\kappa j}\psi_j(x)$, for every $f\in L_2(\Delta,\mu)$; let Γ be a rectifiable closed Jordan curve, positively oriented, such that the disk $|\lambda|\leqq\max[\sup_\nu|\lambda_\nu|, |c|\cdot\|(\beta_{ij})\|]$ is wholly contained in the interior of Γ; and let $f_\lambda(x)$ be the solution of the integral equation $\lambda f_\lambda(x)-(Nf_\lambda)(x)=$

$g(x)$, $(g(x)\in L_2(\Delta,\mu),\ \lambda\in\Gamma)$. Then, for the ordinary part $R_\omega(\lambda)$ of any $S_\omega(\lambda)\in\mathfrak{F}(M)$ and for almost every $x\in\Delta$,

$$(29)\qquad \frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)f_\lambda(x)\,d\lambda=\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)\int_\Delta g(y)\overline{\varphi_\nu(y)}d\mu(y)\cdot\varphi_\nu(x)$$
$$+\sum_{\kappa=1}^{\infty}\sum_{n\geqq 0}a_\omega^{(n)}c^n\int_\Delta g(y)\overline{\psi_\kappa(y)}d\mu(y)\cdot\Psi_\kappa^{(n)}(x),$$

where the expansion of $R_\omega(\lambda)$ is given by $\sum_{n\geqq 0}a_\omega^{(n)}\lambda^n$ on the domain $\{\lambda: |\lambda|<\infty\}$, $\Psi_\kappa^{[n]}(x)=\sum_{j=1}^{\infty}\beta_{\kappa j}^{(n)}\psi_j(x)$, $\Psi_\kappa^{[1]}(x)=\Psi_\kappa(x)$, and $\Psi_\kappa^{[0]}(x)=\psi_\kappa(x)$. If, moreover, $\widetilde{R}_\omega(|\lambda|)$ denotes $\sum_{n\geqq 0}|a_\omega^{(n)}||\lambda|^n$, the series on the right-hand side of (29) is a function in $L_2(\Delta,\mu)$ and its norm never exceeds $\widetilde{R}_\omega(\|N\|)\left\{\int_\Delta|g(x)|^2d\mu(x)\right\}^{\frac{1}{2}}$.

Proof. Since $f_\lambda(x)=[(\lambda I-N)^{-1}g](x)$, the former half of the present corollary is an obvious consequence of Theorem 28. Since, moreover, the operator

$$\frac{1}{2\pi i}\int_\Gamma S_\omega(\lambda)(\lambda I-N)^{-1}d\lambda=R_\omega(N)$$

is a bounded normal operator on $L_2(\Delta,\mu)$ in accordance with the preceding theorem, the series on the right of (29) belongs, of course, to $L_2(\Delta,\mu)$, and its norm is given by $\|[R_\omega(N)g](x)\|$. The final quantity here satisfies the chain of inequalities

$$\|[R_\omega(N)g](x)\|\leqq\sum_{n\geqq 0}|a_\omega^{(n)}|\,\|(N^ng)(x)\|\leqq\widetilde{R}_\omega(\|N\|)\|g(x)\|.$$

The corollary has thus been proved.

Remark. Let $P_\omega(\lambda)$ and $Q_\omega(\lambda)$ be the first and the second principal parts of any $S_\omega(\lambda)\in\mathfrak{F}(M)$ as before. Then, as in Corollary 4, it is also shown that

$$\int_\Gamma P_\omega(\lambda)f_\lambda(x)d\lambda=\int_\Gamma Q_\omega(\lambda)f_\lambda(x)d\lambda=0$$

for almost every x on Δ. Let $\sigma(N)$ and $\{K(\lambda)\}$ denote the continuous spectrum and the complex spectral family of N in Corollary 5 respectively. It can then be verified without difficulty that, for almost every x on Δ,

$$\int_{\{\lambda_\nu\}}R_\omega(\lambda)dK(\lambda)=\sum_{\nu=1}^{\infty}R_\omega(\lambda_\nu)\varphi_\nu(x)\otimes L_{\varphi_\nu(x)}$$

and

$$\int_{\sigma(N)}R_\omega(\lambda)dK(\lambda)=\sum_{\kappa=1}^{\infty}\sum_{n\geqq 0}a_\omega^{(n)}c^n\Psi_\kappa^{[n]}(x)\otimes L_{\psi_\kappa(x)}.$$

Suppose next that $M<|c|\cdot\|(\beta_{ij})\|$. Then the solution $f_\lambda(x)$ of the given equation $\lambda f_\lambda(x)-(Nf_\lambda)(x)=g(x)$, $(g(x)\in L_2(\Delta,\mu))$, possesses the following properties for almost every $x\in\Delta$.

1°. If both (g,φ_ν) and (g,ψ_κ) never vanish for every $\nu,\kappa=1,2,$

$3, \cdots$, there exists at least one (linear or planar) connected set $\mathfrak{D}$ of points (that is, a subset of $\sigma(N)$) in the annular domain $\{\lambda: M \leqq |\lambda| \leqq |c| \cdot \|(\beta_{ij})\|\}$ such that besides every point belonging to the closure of $\{\lambda_\nu\}$, every point in $\mathfrak{D}$ is also a singularity of the function $f_\lambda(x)$ of λ.

2°. A necessary and sufficient condition that the function $f_\lambda(x)$ of λ be regular at an isolated point λ_ν in the closure of $\{\lambda_\nu\}$ is that $g(x)$ be orthogonal to the eigenspace of N corresponding to the eigenvalue λ_ν.

3°. If (g, ψ_κ) vanishes for every $\kappa=1, 2, 3, \cdots$, and if (g, φ_ν) never vanishes for every $\nu=1, 2, 3, \cdots$, then the set of all singularities of the function $f_\lambda(x)$ of λ consists of the closure of $\{\lambda_\nu\}$ alone.

Lastly I add to the bounded normal operator $\tilde{N}[I-K(\varDelta^-(\tilde{N}))]$ in the preceding paper [cf. Proc. Japan Acad., Vol. 40, No. 5, p. 317 (1964)], as follows: its point spectrum is given by $\{\lambda_\nu\}$ or by $\{0\} \cup \{\lambda_\nu\}$ and hence its continuous spectrum is given by $\varDelta^+(\tilde{N})$ or by $\varDelta^+(\tilde{N})-\{0\}$.