81. On a Definition of Singular Integral Operators. I

By Hitoshi KUMANO-GO
Department of Mathematics, Osaka University
(Comm. by Kinjirô KUNUGI, M.J.A., June 12, 1964)

Introduction. The theory of singular integral operators of A. P. Calderón and A. Zygmund [1] has been applied to the various problems in partial differential equations, since A. P. Calderón [2] succeeded in proving the general theorem for the uniqueness of solutions of the Cauchy problem by using this theory. S. Mizohata in the notes [7], [8], and [9] proved the many interesting theorems for the uniqueness by modifying the notion of singular integral operators, M. Yamaguti [12] applied these operators to the existence theorem of solutions of the Cauchy problem for hyperbolic differential equations and M. Matsumura [6] applied to the existence and non-existence theorems of local solutions of the general equations.

In the note [4] we introduced singular integral operators of class $C_{\mathfrak{m}}^{m}$ and proved the theorems of [7] and [8] by a unified method, and also in [5] we generalized the theorem of [9] by applying the operators of this class.

In the present note we shall give a definition of singular integral operators which governs operators of class $C_{\mathfrak{m}}^{m}$, and prove that the main theorems relating to operators of [1] hold for the present operators. In this theory we do not require the homogeneity of the symbol $\sigma(H)(x, \eta)$ in η (Definition 4), but assume the analyticity in η. The technique of almost all the proofs is based on [10] and [12], and the exposition is self-contained. I thank here my colleague K. Ise for helpful discussions.

1. Definitions and lemmas. Let $x=(x_1,\cdots,x_n)$ be a point of Euclidean n-space R_x^n, $\xi=(\xi_1,\cdots,\xi_n)$ be a point of its dual space E_ξ^n and $\alpha=(\alpha_1,\cdots,\alpha_n)$ denote a real vector whose elements are non-negative integers.

We shall use the notations:

$\alpha!=\alpha_1!\cdots\alpha_n!$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $x\cdot\xi=x_1\xi_1+\cdots+x_n\xi_n$,

$D_x=(D_{x_1},\cdots,D_{x_n})=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$, $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $D_\xi=(\cdots$, etc.

The terminology employed is that of L. Schwarz [11].

The Fourier transform $\mathfrak{F}[u](\xi)=\hat{u}(\xi)$ of a function $u\in L_x^2$ is defined by

$$\mathfrak{F}[u](\xi)=\frac{1}{\sqrt{2\pi}^n}\int e^{-\sqrt{-1}x\cdot\xi}u(x)\,dx.$$

We have, then, for $u \in \mathcal{S}_x$[1] and $\gamma \in \mathcal{S}'_x$

$$\widehat{\gamma * u} = (2\pi)^{n/2} \hat{u} \hat{\gamma}, \tag{1.1}$$

and for $a(x) \in \mathcal{B}_x$ the expansion

$$a(y) = \sum_{1 \leqq |\alpha| \leqq l-1} \frac{(y-x)^\alpha}{\alpha !} D_x^\alpha a(x) + \sum_{|\alpha|=l} (y-x)^\alpha a_\alpha(x, y) \tag{1.2}$$

where $a_\alpha(x, y) \in \mathcal{B}(R_x^n \times R_y^n)$ and

$$|D_x^\beta D_y^{\beta'} a_\alpha(x, y)| \leqq C_k \sum_{|\beta''|=k} \sup_x |D_x^{\beta''} a(x)|, \quad k = |\alpha| + |\beta| + |\beta'|. \tag{1.3}$$

Definition 1. *We call a distribution $\lambda \in \mathcal{S}'$ is of type (ρ, τ), ρ, $\tau > 0$, if $\hat{\lambda}(\xi)$ is a function which is positive and infinitely differentiable in $E_\xi^n - \{0\}$, and satisfies*

$$\text{i)} \quad |\xi|^{1/\tau} \leqq C(\hat{\lambda}(\xi)+1) \leqq C'(|\xi|^{1/\rho}+1)$$
$$\text{ii)} \quad |D_\xi^\alpha \hat{\lambda}(\xi)| \leqq C_\alpha \hat{\lambda}(\xi)^{1-\rho|\alpha|} \quad \text{for } |\xi| \geqq 1. \tag{1.4}$$

Remark. If $\hat{\lambda}(\xi)$ is bounded in a neighborhood of the origin, then the second inequality of i) is derived from ii) by setting $|\alpha|=1$.

Now we define a Hilbert space $\mathfrak{H}_p$ $(-\infty < p < +\infty)$ by

$$\mathfrak{H}_p = \{u \in \mathcal{S}'; \hat{u}, \text{ function } \|u\|_p^2 = \int (1+\hat{\lambda}(\xi))^{2p} |\hat{u}(\xi)|^2 d\xi < \infty\}. \tag{1.5}$$

Clearly $\mathfrak{H}_0 = L^2$. In this case we write $\|u\|_0 = \|u\|_{L^2}$ or simply $\|u\|$.

Definition 2. *A convolution operator $\Gamma : \mathcal{S}_x \xrightarrow[\text{into}]{} \mathcal{S}_x$ is called of class $\boldsymbol{T}(p) = \boldsymbol{T}(p, \lambda)$, $-\infty < p < +\infty$, if Γ is defined by $\Gamma u = \gamma * u$, $u \in \mathcal{S}$, where $\gamma \in \mathcal{S}'$ and $\hat{\gamma}$ satisfies*

$$\text{i)} \quad \operatorname{supp} \hat{\gamma}(\xi)^{2)} \subset E_\xi^n - \{0\}$$
$$\text{ii)} \quad \hat{\gamma}(\xi) \in C^\infty(E_\xi^n) \text{ and } |D_\xi^\alpha \hat{\gamma}(\xi)| \leqq C_{\gamma,\alpha} \hat{\lambda}(\xi)^{p-\rho|\alpha|} \text{ for } \xi \neq 0. \tag{1.6}$$

Then, by (1.1) *we can write*

$$\Gamma u = \int e^{\sqrt{-1}\, x \cdot \xi} \hat{\gamma}(\xi) \hat{u}(\xi) d\xi, \quad u \in \mathcal{S}_x. \tag{1.7}$$

Definition 3. *A convolution operator Λ^σ $(\sigma \geqq 0)$ associated with λ is defined by*

$$\Lambda^\sigma u = \int e^{\sqrt{-1}\, x \cdot \xi} \hat{\lambda}(\xi)^\sigma \hat{u}(\xi) d\xi, \quad u \in \mathfrak{H}_\sigma. \tag{1.8}$$

Next we assume there exists a transformation

$$T_s : E_\xi^n \ni \xi = (\xi_1, \cdots, \xi_n) \longrightarrow \eta = (\eta_1, \cdots, \eta_s) \in E_\eta^s \text{ for } \xi \neq 0$$

such that $\eta_j(\xi)$, $j = 1, \cdots, s$ are bounded functions belonging to C^∞ in $E_\xi^n - \{0\}$ and satisfy

$$|D_\xi^\alpha \eta_j(\xi)| \leqq C_\alpha \hat{\lambda}(\xi)^{-\rho|\alpha|} \quad \text{for} \quad |\xi| \geqq 1. \tag{1.9}$$

We must remark, in general $s \neq n$.

Lemma 1. *For $\Gamma \in \boldsymbol{T}(p)$ we define $D_x^\beta x^\alpha \Gamma$ by $(D_x^\beta x^\alpha \Gamma)u = (D_x^\beta x^\alpha \gamma) * u$, $u \in \mathcal{S}_x$. Then we have*

---

1) $\mathcal{S}_x$ denotes the class of rapidly decreasing functions, $\mathcal{S}'_x$ the class of distributions on $\mathcal{S}_x$, and $\mathcal{B}_x$ the class of infinitely differentiable functions whose derivatives are all bounded.

2) For a function $u(x)$, supp $u=$ the closure of $\{x; u(x) \neq 0\}$.

i) $x^\alpha \Gamma \in \boldsymbol{T}(p-\rho|\alpha|)$

ii) *If* $p \geqq 0$ *and* $k \geqq p/(2\rho)$, *then* $(1-\Delta)^{-k}\Lambda^p = \Lambda^p(1-\Delta)^{-k}$ *is extended to a bounded operator in* L^2_x.

iii) *If* $|\alpha| > \{(n+|\beta|)\tau + p\}/\rho$, *then* $D^\beta_x x^\alpha \gamma$ *is a function of* L^1_x *and we have*

$$\|D^\beta_x x^\alpha \gamma\|_{L^1_x} \leqq C_{n,\beta} \operatorname*{Max}_{|\alpha'|\leqq n+2} \|(1+|\xi|)^{|\beta|} D^{\alpha'+\alpha}_\xi \hat{\gamma}(\xi)\|_{L^1_\xi}. \tag{1.10}$$

We can, therefore, extend $D^\beta_x x^\alpha \Gamma$ *to a bounded operator in* L^2_x *and have*

$$\|(D^\beta_x x^\alpha \Gamma)u\|_{L^2} \leqq \|D^\beta_x x^\alpha \gamma\|_{L^1} \cdot \|u\|_{L^2}. \tag{1.11}$$

Proof. i) and ii) are clear by (1.6) and (1.4). iii) As $D^\beta_x x^\alpha \gamma = (2\pi)^{-n/2}\int e^{\sqrt{-1}\,x\cdot\xi}\sqrt{-1}^{|\alpha|+|\beta|}\xi^\beta D^\alpha_\xi \hat{\gamma}(\xi)\,d\xi$, we have

$$\begin{aligned} I(x) &\equiv |(1+|x|^2)^{[n/2+1]} D^\beta_x x^\alpha \gamma| \\ &\leqq (2\pi)^{-n/2}\int |(1-\Delta_\xi)^{[n/2+1]}\{\xi^\beta D^\alpha_\xi \hat{\gamma}(\xi)\}|\,d\xi \\ &\leqq C_{n,\beta} \operatorname*{Max}_{|\alpha'|\leqq n+2} \int (1+|\xi|)^{|\beta|} |D^{\alpha'+\alpha}_\xi \hat{\gamma}(\xi)|\,d\xi. \end{aligned} \tag{1.12}$$

From ii) of (1.6) and i) of (1.4) we have

$$|D^{\alpha'+\alpha}_\xi \hat{\gamma}(\xi)| \leqq C_{\gamma,\alpha',\alpha}|\xi|^{\{p-\rho(|\alpha'|+|\alpha|)\}/\tau}.$$

As $\{p-\rho(|\alpha'|+|\alpha|)\}/\tau < -n-|\beta|$ by the assumption,

$$(1+|\xi|)^{|\beta|}|D^{\alpha'+\alpha}_\xi \hat{\gamma}(\xi)| \in L^1_\xi.$$

Hence $I(x)$ is bounded and this shows $D_x x^\alpha \gamma \in L^1_x$. From a well-known formula

$$\|f*g\|_{L^p} = \|f\|_{L^1}\cdot\|g\|_{L^p},\ f\in L^1,\ g\in L^p \quad (p\geqq 1) \tag{1.13}$$

we get (1.11).

Now, for $\eta^{(0)} \in E^s_\eta$ and a positive number δ we denote

$$\mathcal{D}(\eta^{(0)},\delta) = \{\eta \in E^s_\eta;\ |\eta_j - \eta^{(0)}_j| < \delta,\ j=1,\cdots,s\},$$
$$\mathcal{D}^*(\eta^{(0)},\delta) = \{\zeta \in \boldsymbol{C}^s_\zeta;\ |\zeta_j - \eta^{(0)}_j| < \delta,\ j=1,\cdots,s\}$$

where $\boldsymbol{C}^s_\zeta$ $(\supset E^s_\eta)$ denote a complex s-dimensional space.

Definition 4. *We call* H *a singular integral operator of class* $\boldsymbol{S}(\lambda, T_s)$ *with the symbol* $\sigma(H)(x,\eta)$, *if the following conditions are satisfied.*

There exist positive numbers $\delta < \delta'$ *and* $\eta^{(i)} \in E^s$ $(i=1,\cdots,k)$ *for some* k *such that* $\sigma(H)(x,\eta)$ *is written as*

$$\sigma(H)(x,\eta) = \sum_{i=1}^k h_i(x,\eta)\alpha_i(\eta)$$

where $\alpha_i(\eta) \in C^\infty_0$ *in* $\mathcal{D}(\eta^{(i)},\delta)$ *and* $h_i(x,\eta)$ *are extended to functions of* $\mathcal{B}$ *in* $R^n_x \times \mathcal{D}^*(\eta^{(i)},\delta')$ *and analytic in* $\mathcal{D}^*(\eta^{(i)},\delta')$ *for any fixed* $x \in R^n_x$. *Then,* Hu *is defined by*

$$Hu = \frac{1}{\sqrt{2\pi}^n}\int e^{\sqrt{-1}\,x\cdot\xi}\sigma(H)(x,\eta(\xi))\hat{u}(\xi)\,d\xi,\ u \in L^2_x.$$

Since $h_i(x,\eta)$ $(i=1,\cdots,k)$ are analytic in η, by Cauchy's formula we can extend it as

$$h_i(x,\eta) = \sum_\nu a^{(\nu)}_i(x)(\eta-\eta^{(i)})^\nu,\ \nu=(\nu_1,\cdots,\nu_s)$$

where

$$(1.14)\qquad |a_i^{(\nu)}(x)(\eta-\eta^{(i)})^\nu| \leqq \sup_{R^n\times\mathcal{D}^*(\eta^{(i)},\delta')} |h_i(x,\zeta)|\cdot\left(\frac{\delta}{\delta'}\right)^{|\nu|}$$

$$\text{for } (x,\eta)\in R^n\times\mathcal{D}(\eta^{(i)},\delta).$$

Hence, if we define convolution operators $H_i^{(\nu)}$ by

$$(1.15)\qquad \widehat{H_i^{(\nu)}u}(\xi)=h_i^{(\nu)}(\eta(\xi))\hat{u}(\xi) \quad \text{where} \quad h_i^{(\nu)}(\eta)=(\eta-\eta^{(i)})^\nu\alpha_i(\eta),$$

we can write Hu as

$$(1.16)\qquad Hu=\sum_{i=1}^{k}\sum_{\nu} a_i^{(\nu)}H_i^{(\nu)}u,$$

and also by (1.9) we have

$$(1.17)\qquad H_i^{(\nu)}\Gamma=\Gamma H_i^{(\nu)}\in \boldsymbol{T}(p) \quad \text{for} \quad \Gamma\in\boldsymbol{T}(p).$$

Definition 5. *Let R_1 and R_2 be bounded operators in L_x^2. We write $R_1\overset{\theta}{\equiv}R_2$, $\theta>0$, if for any $\Gamma\in\boldsymbol{T}(p)$ $(-\infty<p<+\infty)$ and $\sigma_0\geqq 0$ we can write*

$$\Gamma(R_1-R_2)=\sum_{j=1}^{l} H_j\Gamma_j+K_{\sigma_0}$$

$$(R_1-R_2)\Gamma=\sum_{j=1}^{l'} H_j'\Gamma_j'+K_{\sigma_0}'$$

for sufficiently large l and l' depending on Γ and $\sigma_0\geqq 0$, where H_j, $H_j'\in\boldsymbol{S}(\lambda, T_s)$, Γ_j, $\Gamma_j'\in\boldsymbol{T}(p-\theta)$, and K_{σ_0}, K_{σ_0}' are bounded operators of order σ_0.[3] *If we can take $l=l'=0$ for any Γ and $\sigma_0\geqq 0$, we write $R_1\overset{\infty}{\equiv}R_2$.*

Lemma 2. *Let Ψ be a bounded operator in L_x^2 defined by $\widehat{\Psi u}(\xi)=\psi(\xi)\hat{u}(\xi)$ where $\psi(\xi)$ is a bounded function which has compact support. Then, $\Psi\overset{\infty}{\equiv}0$.*

Proof. It is clear as $\Lambda^{\sigma_1}\Gamma\Psi\Lambda^{\sigma_2}$ and $\Lambda^{\sigma_1}\Psi\Gamma\Lambda^{\sigma_2}$ are bounded operators in L_x^2 for any σ_1, $\sigma_2\geqq 0$ and $\Gamma\in\boldsymbol{T}(p)$.

Lemma 3. *Let $a(x)\in\mathcal{B}_x$ and $\Gamma\in\boldsymbol{T}(p)$, $-\infty<p<+\infty$. Then for any $\sigma_0\geqq 0$ we have the representation*

$$(1.18)\qquad \begin{aligned}\Gamma a-a\Gamma&=\sum_{1\leqq|\alpha|\leqq l-1}\frac{(-1)^{|\alpha|}}{\alpha!}D_x^\alpha a\cdot(x^\alpha\Gamma)+K_{\sigma_0}^{(1)}\\ &=-\sum_{1\leqq|\alpha|\leqq l-1}\frac{1}{\alpha!}(x^\alpha\Gamma)D_x^\alpha a+K_{\sigma_0}^{(2)}\end{aligned}$$

for every $l>\mathrm{Max}\,[\{(4k+n)\tau+p\}/\rho, 0]$ with $k=[\sigma_0/(2\rho)+1]$, where $K_{\sigma_0}^{(i)}$ $(i=1,2)$ are of order σ_0 and

$$(1.19)\qquad \begin{aligned}&\|\Lambda^{\sigma_1}K_{\sigma_0}^{(i)}\Lambda^{\sigma_2}\|\\ &\leqq C_{\sigma_0,l}\max_{l\leqq|\alpha|\leqq l+n+2}\|(1+|\xi|)^{4k}D_\xi^\alpha\hat{\gamma}(\xi)\|_{L^1}\max_{|\beta|\leqq 4k+l}|D_x^\beta a|\\ &\qquad (i=1,2,\ 0\leqq\sigma_1,\ \sigma_2\leqq\sigma_0).\end{aligned}$$

Proof. Using (1.2) and remarking $x^\alpha\gamma\in L_x^1$ for $|\alpha|=l$ we have for $u\in\mathcal{S}_x$

3) An operator K in L_x^2 is called of order σ_0, if $\Lambda^{\sigma_1}K\Lambda^{\sigma_2}$ $(0\leqq\sigma_1,\ \sigma_2\leqq\sigma_0)$ are bounded operators in L_x^2. We denote such an operator by K_{σ_0} with a suffix σ_0.

$$(\Gamma a - a\Gamma)u = \gamma_y(\{a(x-y)-a(x)\}u(x-y))$$
$$= \sum_{1\leqq|\alpha|\leqq l-1} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha a(x)\cdot(y^\alpha\gamma_y)(u(x-y))$$
$$+ (-1)^l \sum_{|\alpha|=l} \int (x^\alpha\gamma)(x-y)\cdot a_\alpha(x, x-y)u(y)\,dy$$
$$\equiv I_1u + I_2u.$$

It is easy to see

$$I_1u = \sum_{1\leqq|\alpha|\leqq l-1} \frac{(-1)^{|\alpha|}}{\alpha!} D_x^\alpha a\cdot(x^\alpha\Gamma)u.$$

For $k=[\sigma_0/(2\rho)+1]$ if we write $\Lambda^{\sigma_1}I_2\Lambda^{\sigma_2}u$ as

$$\Lambda^{\sigma_1}I_2\Lambda^{\sigma_2}u = \Lambda^{\sigma_1}(1-\Delta)^{-k}\{(1-\Delta)^k I_2(1-\Delta)^k\}(1-\Delta)^{-k}\Lambda^{\sigma_2}u,$$

then, by ii) of Lemma 1 we may only prove the boundedness of $J=(1-\Delta)^k I_2(1-\Delta)^k$.

$$|(Ju)(x)|$$
$$\leqq \left|(1-\Delta_x)^k \sum_{|\alpha|=l} \int (x^\alpha\gamma)(x-y)\cdot a_\alpha(x, x-y)(1-\Delta_y)^k u(y)\,dy\right|$$
$$= \left|\int (1-\Delta_x)^k(1-\Delta_y)^k\{(x^\alpha\gamma)(x-y)\cdot a_\alpha(x, x-y)\}u(y)dy\right|$$
$$\leqq C_k \operatorname*{Max}_{|\beta'|+|\beta''|=4k} \sup_{x,y} |D_x^{\beta'}D_y^{\beta''}a_\alpha(x,y)| \operatorname*{Max}_{|\beta|\leqq 4k} \int |(D_x^\beta x^\alpha\gamma)(x-y)|\ |u(y)|\,dy.$$

Here we remark $D_x^\beta x^\alpha\gamma \in L_x^1$ by iii) of Lemma 1 and

$$|\alpha| = l > \{(4k+n)\tau + p\}/\rho.$$

Hence we have by (1.3) and (1.13)

$$||(Ju)(x)||_{L^2} \leqq C_{k,l} \operatorname*{Max}_{|\beta|\leqq 4k+l} |D_x^\beta a|\cdot \operatorname*{Max}_{|\beta|\leqq 4k} ||D_x^\beta x^\alpha\gamma||_{L^1}\cdot||u||_{L^2}.$$

This shows that the first equality of (1.18) holds. The second is obtained, if we expand $(a(x-y)-a(x))$ with the base $(x-y)$. Q.E.D.

Let $\{g^0\}$ be the set of lattice points in R_x^n and $\{a_{g^0}\}$ a set of functions of $\mathcal{B}_x$ such that

$$a_{g^0}(x) \in C_0^\infty(\mathcal{D}_{g^0,\delta}),\ |D_x^\alpha a_{g^0}(x)| \leqq A_{0,k} \quad \text{for} \quad |\alpha|\leqq k$$

where $\delta>0$ is a fixed constant and

$$\mathcal{D}_{g^0,\delta} = \{x; |x-g^0| < \delta\}.$$

If we set $g=\varepsilon/\delta g^0$ and $a_g(x)=a_{g^0}(\delta/\varepsilon x)$, then

(1.20) $$a_g(x) \in C_0^\infty(\mathcal{D}_{g,\delta}),\ |D_x^\alpha a_g(x)| \leqq A_k\varepsilon^{-|\alpha|} \quad \text{for} \quad |\alpha|\leqq k.$$

Lemma 4 (S. Mizohata). *Let* $\{a_g\}$ *be a set of functions of* (1.20) *and* $\Gamma\in \boldsymbol{T}(p)$ *where* $0<p\leqq\rho$.

Then we have for every $0<\varepsilon<1$

(1.21) $$\sum_g ||(\Gamma a_g - a_g\Gamma)u||_{L^2}$$
$$\leqq C_l\varepsilon^{-2l}A_l^2\{\sum_{1\leqq|\alpha|\leqq l-1} \sup_\xi |D_\xi^\alpha\hat{\gamma}(\xi)|^2 + \sum_{|\alpha|=l} ||x^\alpha\gamma||_{L^1}^2\}||u||_{L^2}^2$$

where $l=2\,\mathrm{Max}\,\{[(n\tau+p)/(2\rho)+1],\ [n/4+1]\}$.

Proof. Set $I_gu=(\Gamma a_g - a_g\Gamma)u$. Then, by the similar way as the proof of Lemma 3 we have

(2.22) $$|(I_gu)(x)|$$
$$\leqq C_l\varepsilon^{-2l}A_l^2\{\sum_{1\leqq|\alpha|\leqq l-1} |(x^\alpha\Gamma)u(x)|^2 + \sum_{|\alpha|=l} (\int |(x^\alpha\gamma)(x-y)|\,|u(y)|\,dy)^2.$$

Here we remark $x^\alpha\gamma \in L^1_x$ by iii) of Lemma 1 and $|D^\alpha_x a_g| \leqq A_l \varepsilon^{-l}$ for $|\alpha| \leqq l$. If $x \in \mathcal{D}_{g,2\varepsilon}$, we have $a_g(x-y)=0$ for $|y| \leqq |x-g|/2$.
Hence $a_g(x-y)/|y|^l \in C_0^\infty(R^n_y)$, so that we have

$$|(I_g u)(x)| = |(\Gamma a_g u)(x)| = |\gamma_y(|y|^l a_g(x-y)/|y|^l \cdot u(x-y))|$$
$$\leqq C'_l \frac{A_0}{|x-g|^l} \int |(|x|^l\gamma)(x-y)| \, |u(y)| dy, \ x \notin \mathcal{D}_{g,2\varepsilon}.$$

Here we remark $|x|^l$ is a polynomial, as l is even number. Since $\sum_{g;|x-g|\geqq\varepsilon} |x-g|^{-2l} = C_n \varepsilon^{-(2l-n)}$ for $2l>n$, we have

$$(1.23) \qquad \sum_{g;\, x \notin \mathcal{D}_{g,2\varepsilon}} |(I_g u)(x)|^2 \leqq C''_l A_0^2 \varepsilon^{-2l} \{\int (|x|^l\gamma)(x-y)| \, |u(y)| dy\}^2.$$

As the number of g such that $x \in \mathcal{D}_{g,2\varepsilon}$ is finite and independent of ε, we see from (1.22) and (1.23) that (1.22) holds even if we replace $|(I_g u)(x)|^2$ by $\sum_g |(I_g u)(x)|^2$. Then, using $\sup_\xi |\widehat{x^\alpha\gamma}(\xi)| = \sup_\xi |D^\alpha_\xi \hat{\gamma}(\xi)| < \infty$ for $0<p\leqq\rho$ and (1.13), we get (1.21). Q.E.D.

(References are listed at the end of the next article, p. 378.)