139. On Differentiability in Time of Solutions of Some Type of Boundary Value Problems

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(Comm. by Kinjirô KUNUGI, M.J.A., Oct. 12, 1964)

1. Introduction. The differentiability problem of the solutions of the abstract differential equation

$$du(t)/dt+A(t)u(t)=f(t)$$

in a Banach space was treated by S. Agmon and L. Nirenberg ([2]) quite generally when $A(t)$ does not depend on t. A. Friedman [4] generalized some of their results to the equations in a Hilbert space in which $A(t)$ may depend on t. However he assumes that the domain of $A(t)$ does not depend on t, therefore his theorem cannot be applied directly to the boundary value problem

$$\partial u(t,x)/\partial t+A(t,x,D_x)u(t,x)=f(t,x), \quad x\in\Omega, \tag{0.1}$$

$$B_j(t,x,D_x)u(t,x)=0, \quad x\in\partial\Omega, \quad j=1,\cdots,m, \tag{0.2}$$

where $D_x=(\partial/\partial x_1,\cdots,\partial/\partial x_n)$ and $A(t,x,D_x)$ is an elliptic operator of order $2m$ in a bounded domain $\Omega\subset R^n$ for each t, unless the coefficients of $B_j(t,x,D_x)$, $j=1,\cdots,m$, are independent of t. The object of the present note is to show that A. Friedman's method can be applied to the problem (0.1)–(0.2) when the positive and negative imaginary axes are of minimal growth with respect to the system $A(t,x,D_x)$, $\{B_j(t,x,D_x)\}_{j=1}^m$ in the sense of S. Agmon [1], and hence that the solution of (0.1)–(0.2) is smooth in t as a function with values in $L^2(\Omega)$ or $H_{2m}(\Omega)$ if $f(t,x)$ and the coefficients of $A(t,x,D_x)$, $B_j(t,x,D_x)$, $j=1,\cdots,m$, are sufficiently smooth.

2. Preliminary lemmas. Let Ω be a bounded domain with a smooth boundary in R^n. By $H_k(\Omega)$ we denote the set of all measurable functions in Ω whose distribution derivatives of order up to k are square integrable, the norm of $H_k(\Omega)$ being denoted by $\|\ \|_k$.

Assumptions. (I) For each $t\in(-\infty,\infty)$ $A(t,x,D_x)=\sum_{|\alpha|\leq 2m}a_\alpha(t,x)D_x^\alpha$ is an elliptic operator of order $2m$ in $\bar{\Omega}$.

(II) $\{B_j(t,x,D_x)\}_{j=1}^m=\{\sum_{|\beta|\leq m_j}b_{j\beta}(t,x)D_x^\beta\}_{j=1}^m$ is a normal system of boundary operators for each t. The order m_j of $B_j(t,x,D_x)$ is smaller than $2m$ and does not depend on t.

(III) $$\pm(-1)^m iD_y^{2m}-A(t,x,D_x) \tag{1.1}$$

is elliptic with respect to (x,y) in the cylindrical domain $\Omega\times\{y;-\infty<y<\infty\}$ for each fixed t. The Complementing Condition is satisfied by (1.1) and $\{B_j(t,x,D_x)\}_{j=1}^m$ in $\Omega\times\{y;-\infty<y<\infty\}$ for each t.

(IV) The coefficients of $A(t,x,D_x)$ as well as those of $\{B_j(t,x,$

$D_x)\}_{j=1}^m$ which may be supposed to be defined in the whole of $\bar{\Omega}\times\{t; -\infty<t<\infty\}$ are sufficiently smooth.

LEMMA 1.1. *Under the assumptions* (I)–(IV) *there exists a positive number* N *such that if* $\lambda>N$ *or* $\lambda<-N$

$$\sum_{k=0}^{2m}|\lambda|^{\frac{2m-k}{2m}}\|u\|_k\leqq C_1\{\|(i\lambda+A(t,x,D_x))u\|_0+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda|^{\frac{2m-m_j-k}{2m}}\|w_j\|_k\}$$

for each $u\in H_{2m}(\Omega)$ *where* w_j *is an arbitrary function in* $H_{2m-m_j}(\Omega)$ *which coincides with* $B_j(t,x,D_x)u(x)$ *on the boundary of* Ω. *If* $|\lambda|\leqq N$ *we have*

$$\|u\|_{2m}\leqq C_2\{\|(i\lambda+A(t,x,D_x))u\|_0+\sum_{j=1}^{m}\|w_j\|_{2m-m_j}+\|u\|_0\}.$$

The above lemma is a consequence of Agmon-Douglis-Nirenberg inequality applied to (1.1), $\{B_j(t,x,D_x)\}$ and the function $\zeta(y)e^{iy\mu}u(x)$ where $\zeta(y)$ is a real valued function satisfying $\zeta(y)=1$ near the origin and having a compact support and μ is a real number (cf. S. Agmon [1]).

LEMMA 1.2. *Suppose* f *and* g *are complex valued functions of a real variable* λ *with* $f\in L^1(-\infty,\infty)$ *and* $g\in L^2(-\infty,\infty)$. *Then for* $0<l<1$, *we have*

$$\sqrt{\int_{-\infty}^{\infty}(|\lambda|^l|(f*g)(\lambda)|)^2d\lambda}$$
$$\leqq\int_{-\infty}^{\infty}|\lambda|^l|f(\lambda)|d\lambda\sqrt{\int_{-\infty}^{\infty}|g(\lambda)|^2d\lambda}+\int_{-\infty}^{\infty}|f(\lambda)|d\lambda\sqrt{\int_{-\infty}^{\infty}(|\lambda|^l|g(\lambda)|)^2d\lambda}.$$

3. Main theorem. Let $v(t,x)$ be a function with values in $H_{2m}(\Omega)$ in $-\infty<t<\infty$ and be a solution of the boundary value problem

$$\partial v(t,x)/\partial t+A(t,x,D_x)v(t,x)=f(t,x),\quad -\infty<t<\infty,\quad x\in\Omega, \tag{2.1}$$

$$B_j(t,x,D_x)v(t,x)=g_j(t,x),\quad -\infty<t<\infty,\quad x\in\partial\Omega,\quad j=1,\cdots,m, \tag{2.2}$$

where $f(t,x)$ and $g_j(t,x)$, $j=1,\cdots,m$, are functions of t with values in $L^2(\Omega)$ and $H_{2m-m_j}(\Omega)$, $j=1,\cdots,m$, respectively. Furthermore we assume that $v(t,x)\equiv0$ when $|t-s|>\delta$ where s is a fixed real number and δ is a sufficiently small positive number which should be specified later. Let $\varphi(t)$ be a smooth real valued function satisfying

$$\varphi(t)=\begin{cases}1 & \text{if} \quad -1<t<1,\\ 0 & \text{if} \quad |t|>2,\end{cases}$$

and $\psi(t)=\varphi((t-s)/\delta)$. Then

$$\partial v(t,x)/\partial t+A(s,x,D_x)v(t,x)=F(t,x),\quad x\in\Omega,$$
$$B_j(s,x,D_x)v(t,x)=G_j(t,x),\quad j=1,\cdots,m,\quad x\in\partial\Omega,$$

where

$$F(t,x)=f(t,x)+\sum_{|\alpha|\leqq 2m}\psi(t)(a_\alpha(s,x)-a_\alpha(t,x))D_x^\alpha v(t,x)$$
$$G_j(t,x)=g_j(t,x)+\sum_{|\beta|\leqq m_j}\psi(t)(b_{j\beta}(s,x)-b_{j\beta}(t,x))D_x^\beta v(t,x)$$
$$\equiv g_j(t,x)+h_j(t,x).$$

The Fourier transform

$$\hat{v}(\lambda, x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda t} v(t, x)\,dt$$

of $v(t, x)$ with respect to t satisfies

$$i\lambda\hat{v}(\lambda, x)+A(s, x, D_x)\hat{v}(\lambda, x)=\hat{F}(\lambda, x), \quad x\in\Omega,$$
$$B_j(s, x, D_x)\hat{v}(\lambda, x)=\hat{G}_j(\lambda, x), \quad x\in\partial\Omega, \quad j=1, \cdots, m.$$

By Lemma 1.1 we have

$$\sum_{k=0}^{2m}|\lambda|^{\frac{2m-k}{2m}}\|\hat{v}(\lambda)\|_k \leqq C_3\{\|\hat{F}(\lambda)\|_0+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}|\lambda|^{\frac{2m-m_j-k}{2m}}\|\hat{G}_j(\lambda)\|_k\} \quad \text{if } |\lambda|\geqq N,$$

$$\|\hat{v}(\lambda)\|_{2m} \leqq C_4\{\|\hat{F}(\lambda)\|_0+\sum_{j=1}^{m}\|\hat{G}_j(\lambda)\|_{2m-m_j}+\|\hat{v}(\lambda)\|_0\} \quad \text{if } |\lambda|<N.$$

Thus we have

$$\begin{aligned}\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-k}{2m}}\|\hat{v}(\lambda)\|_k)^2 d\lambda \leqq C_5\Big\{\int_{-\infty}^{\infty}\|\hat{F}(\lambda)\|_0^2 d\lambda \\ +\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-m_j-k}{2m}}\|\hat{G}_j(\lambda)\|_k)^2 d\lambda+\int_{-\infty}^{\infty}\|\hat{v}(\lambda)\|_0^2\, d\lambda\Big\}.\end{aligned} \tag{2.1}$$

Writing $\gamma_{j\beta}(t, s, x)=\psi(t)(b_{j\beta}(s, x)-b_{j\beta}(t, x))$ we have for any multi-index κ with $|\kappa|\leqq k$

$$D_x^\kappa h_j(t, x)=\sum_{|\beta|\leqq m_j}\sum_{\nu\leqq\kappa} D_x^{\kappa-\nu}\gamma_{j\beta}(t, s, x) D_x^{\nu+\beta} v(t, x).$$

In order to obtain an estimate of the right side of (2.1), we must estimate

$$\sqrt{\int_{-\infty}^{\infty}\int_{\Omega}(|\lambda|^l|\widehat{D_x^\kappa h_j}(\lambda, x)|)^2\, dx d\lambda}$$

where $l=(2m-m_j-k)/2m$. By Lemma 1.2 we get

$$\begin{aligned}&\sqrt{\int_{-\infty}^{\infty}(|\lambda|^l|\widehat{D_x^{\kappa-\nu}\gamma_{j\beta}(\cdot, s, x)*D_x^{\nu+\beta}\hat{v}(\cdot, x)}(\lambda)|)^2 d\lambda} \\ &\leqq\int|\widehat{D_x^{\kappa-\nu}\gamma_{j\beta}}(\lambda, s, x)|\,d\lambda\sqrt{\int(|\lambda|^l|D_x^{\nu+\beta}\hat{v}(\lambda, x)|)^2 d\lambda} \\ &\quad+\int|\lambda|^l|\widehat{D_x^{\kappa-\nu}\gamma_{j\beta}}(\lambda, s, x)|\,d\lambda\sqrt{\int|D_x^{\nu+\beta}\hat{v}(\lambda, x)|^2\, d\lambda}.\end{aligned} \tag{2.2}$$

It is easy to show that there exists a constant K such that

$$|D_x^{\kappa-\nu}\gamma_{j\beta}(t, s, x)|\leqq K\delta,$$
$$|(\partial/\partial t)^2 D_x^{\kappa-\nu}\gamma_{j\beta}(t, s, x)|\leqq K/\delta,$$

which implies that for any given $\varepsilon>0$ we have

$$\int_{-\infty}^{\infty}|\lambda|^l|\widehat{D_x^{\kappa-\nu}\gamma_{j\beta}}(\lambda, s, x)|\,d\lambda<\varepsilon, \tag{2.3}$$

$$\int_{-\infty}^{\infty}|\widehat{D_x^{\kappa-\nu}\gamma_{j\beta}}(\lambda, s, x)|\,d\lambda<\varepsilon, \tag{2.4}$$

when δ is sufficiently small. Combining (2.1)–(2.4) and estimating the other terms similarly we get

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-k}{2m}}\|\hat{v}(\lambda)\|_k)^2 d\lambda$$
$$\leqq C_6\Big\{\int_{-\infty}^{\infty}\|\hat{f}(\lambda)\|_0^2 d\lambda+\sum_{j=1}^{m}\sum_{k=0}^{2m-m_j}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-m_j-k}{2m}}\|\hat{g}_j(\lambda)\|_k)^2 d\lambda$$
$$+\int_{-\infty}^{\infty}\|\hat{v}(\lambda)\|_0^2\, d\lambda\Big\} \tag{2.5}$$

when δ is sufficiently small.

If $u(t,x)$ is a function of t with values in $H_{2m}(\Omega)$ satisfying

$$\partial u(t,x)/\partial t+A(t,x,D_x)u(t,x)=f(t,x),\quad x\in\Omega,$$
$$B_j(t,x,D_x)u(t,x)=0,\quad x\in\partial\Omega,\quad j=1,\cdots,m,$$

then $v(t,x)=\varphi(2(t-s)/\delta)u(t,x)\equiv\Psi(t)u(t,x)$ is a solution of

$$\partial v(t,x)/\partial t+A(t,x,D_x)v(t,x)$$
$$=\Psi(t)f(t,x)+\Psi'(t)u(t,x)\equiv f_1(t,x),\quad x\in\Omega,$$
$$B_j(t,x,D_x)v(t,x)=0,\quad x\in\partial\Omega,\quad j=1,\cdots,m.$$

Thus by (2.5) and Parseval theorem

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-k}{2m}}\|\hat{v}(\lambda)\|_k)^2 d\lambda$$
$$\leqq C_7\Big\{\int_{-\infty}^{\infty}\|f(t)\|_0^2 dt+\int_{-\infty}^{\infty}\|u(t)\|_0^2\, dt\Big\}. \tag{2.6}$$

If the derivative $\dot{v}(t,x)=\partial v(t,x)/\partial t$ is also a function of t with values in $H_{2m}(\Omega)$, it satisfies

$$\partial\dot{v}(t,x)/\partial t+A(t,x,D_x)\dot{v}(t,x)=\dot{f}_1(t,x)-\dot{A}(t,x,D_x)v(t,x),\quad x\in\Omega,$$
$$B_j(t,x,D_x)\dot{v}(t,x)=-\dot{B}_j(t,x,D_x)v(t,x),\quad x\in\partial\Omega.$$

Since $\dot{B}_j(t,x,D_x)v(t,x)$ is a function of t with values in $H_{2m-m_j}(\Omega)$, we can apply (2.5) to $\dot{v}(t,x)$ and noting (2.6) we get

$$\sum_{k=0}^{2m}\int_{-\infty}^{\infty}(|\lambda|^{\frac{2m-k}{2m}}\|\hat{\dot{v}}(\lambda)\|_k)^2 d\lambda$$
$$\leqq C_8\Big\{\int_{-\infty}^{\infty}\|f(t)\|_0^2 dt+\int_{-\infty}^{\infty}\|\dot{f}(t)\|_0^2\, dt+\int_{-\infty}^{\infty}\|u(t)\|_0^2\, dt\Big\}. \tag{2.7}$$

The left members of (2.6) and (2.7) dominate

$$\int_{-\infty}^{\infty}\Big\|\frac{dv(t)}{dt}\Big\|_0^2 dt+\int_{-\infty}^{\infty}\|v(t)\|_{2m}^2\, dt,$$
$$\int_{-\infty}^{\infty}\Big\|\frac{d^2v(t)}{dt^2}\Big\|_0^2 dt+\int_{-\infty}^{\infty}\Big\|\frac{dv(t)}{dt}\Big\|_{2m}^2 dt$$

respectively. A repeated application of the above argument shows that

THEOREM 1. *The solution of the boundary value problem* (0.1)–(0.2) *is a smooth function of t with values in $L^2(\Omega)$ or $H_{2m}(\Omega)$ if the coefficients of $A(t,x,D_x)$ and $B_j(t,x,D_x)$, $j=1,\cdots,m$, $f(t,x)$ and the boundary of Ω are sufficiently smooth.*

Arguing as in the last section of [4], we can prove

THEOREM 2. *Suppose in addition to the assumptions of Theorem 1 that the coefficients of* $A(t, x, D_x)$ *and* $B_j(t, x, D_x)$, $j=1, \cdots, m$, *as well as some of their derivatives in* x *are uniformly analytic in* t *and that* $f(t, x)$ *is an analytic function of* t *with values in* $L^2(\Omega)$. *Then the solution of* (0.1)–(0.2) *is also an analytic function of* t *with values in* $L^2(\Omega)$ *or* $H_{2m}(\Omega)$.

References

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