137. *A Duality Theorem for the Real Unimodular Group of Second Order*

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Let G be the real special linear group of second order. G consists of all real matrices g such that

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad ad-bc=1.$$

The purpose of the present paper is to characterize G as a "dual" of the space of its irreducible unitary representations. This space is furnished with a law according to which the Kronecker product of any two representations is decomposed into irreducible components.

This duality may be regarded as an analogue of the Tannaka's duality theorem in the case of compact groups.

Let K be a compact group, Tannaka's duality theorem states the following. Consider the totality X of irreducible unitary representations of K. The Kronecker product $\rho\otimes\sigma$ of two elements in X is decomposed into the direct sum $\sum_j\oplus\tau_j$ of finite irreducible representations. In this decomposition, let $u\otimes v$ is equal to $\sum_j\oplus w_j$, in which u, v, w_j, are vectors in the spaces of representations ρ, σ, τ_j respectively. An element k of K decides an operator field over X, which consists of unitary matrices $\rho(k)$ in each space of representation ρ. And the decomposition of the Kronecker product $\rho(k)u\otimes\sigma(k)v$ is equal to $\sum_j\oplus\tau_j(k)w_j$. Conversely let $\{T(\rho)\}$ be an operator field over X, such that each $T(\rho)$ is a unitary matrix in the space of representation ρ, and $(T(\rho)u)\otimes(T(\sigma)v)$ is equal to $\sum_j\oplus(T(\tau_j)w_j)$, for any u and v. The duality theorem affirms that the totality of operator fields as above coincides with the original group K, that is, K is characterized as a "dual" of the space of its irreducible unitary representations, and the initial topology of K corresponds to the weakest topology which makes all the matrix element $\langle T(\rho)u, v\rangle$ continuous.

Our main theorem characterizes G in the same way as for the case of compact groups. Let Ω be the set of all equivalence classes of irreducible unitary (therefore infinite-dimensional) representations of G. We choose and fix a representation $\omega=\{U_g(\omega), \mathfrak{H}(\omega)\}$ of G from each element of Ω. The Kronecker product $\{U_g(\sigma)\otimes U_g(\tau), \mathfrak{H}(\sigma)\otimes\mathfrak{H}(\tau)\}$, in which σ, τ are elements of Ω, is decomposed into irreducible components as follows.

$$u\otimes v=\int_{\Omega} w(\omega)\, d\mu_{\sigma,\tau}(\omega).$$

In this equality, u, v, and $w(\omega)$ are vectors in $\mathfrak{H}(\sigma)$, $\mathfrak{H}(\tau)$, and $\mathfrak{H}(\omega)$ respectively, and $\mu_{\sigma,\tau}$ is a measure on Ω depending only to σ, τ.

Let $\boldsymbol{T}=\{T(\omega)\}$ be a unitary operator field over Ω, $T(\omega)$ being a unitary operator in $\mathfrak{H}(\omega)$.

Such a $\boldsymbol{T}$ is called *admissible*, if for arbitrary u, v,

$$T(\sigma)u\otimes T(\tau)v=\int_{\Omega} T(\omega)w(\omega)\, d\mu_{\sigma,\tau}(\omega).$$

For instance any fixed element g in G gives an admissible unitary operator field $\boldsymbol{U}_g=\{U_g(\omega)\}$.

Now we define the product $\boldsymbol{TS}$ of two admissible operator fields $\boldsymbol{T}=\{T(\omega)\}$ and $\boldsymbol{S}=\{S(\omega)\}$ by

$$TS(\omega)=T(\omega)S(\omega).$$

Then the totality $\mathfrak{R}$ of admissible unitary operator fields becomes a group with this multiplication, and the mapping $g\rightarrow \boldsymbol{U}_g$ gives an homomorphism from G into $\mathfrak{R}$. Our main theorem is stated as follows.

Theorem. *The mapping $g\rightarrow \boldsymbol{U}_g$ is an isomorphism from* G *onto* $\mathfrak{R}$. *And in this correspondence, the natural topology in G coincides with the weakest topology, which makes all the matrix element $\langle U_g(\omega)u, v\rangle$ (ω: fixed) continuous.*

1. Now we recall certain known facts on irreducible unitary representations of G, which we make use to prove the main theorem.

a) V. Bargmann classified all irreducible unitary representations of G to four series and to eight sub-series as follows.

1) Principal series,
   - i) integral (non-spinor) representation, C_l^0 $(1/4\leq l<\infty)$,
   - ii) half-integral (spinor) representation, $C_l^{1/2}$ $(1/4<l<\infty)$.

2) Supplementary series, C_l^0 (or E_l) $(0<l<1/4)$.

3) Discrete series,
   - i) positive non-spinor representation, D_n^+ $(n=1, 2,\cdots)$,
   - ii) positive spinor representation, D_n^+ $(n=1/2, 3/2,\cdots)$,
   - iii) negative non-spinor representation, D_n^- $(n=1, 2,\cdots)$,
   - iv) negative spinor representation, D_n^- $(n=1/2, 3/2,\cdots)$.

4) Identity representation, I.

b) Since the subgroup

$$R=\left\{r(\theta)=\begin{pmatrix}\cos(\theta/2), & -\sin(\theta/2)\\ \sin(\theta/2), & \cos(\theta/2)\end{pmatrix};\ -2\pi<\theta\leqq 2\pi\right\},$$

is abelian and compact, so the restriction of each irreducible representation ω of G to R is decomposed into direct sum of multiples of one-dimensional representations $\rho_k=\{\exp(ik\theta), H_k(\omega)\}$. In this case the multiplicities are at most one for any k, and the index k runs;

$k=0, \pm 1, \pm 2,\cdots$ for 1)— i) and 2),

$k=\pm(1/2), \pm(3/2), \cdots$ for 1)− ii),
$k=n, n+1, n+2, \cdots$ for 3)− i) and 3)− ii),
$k=-n, -n-1, -n-2, \cdots$ for 3)−iii) and 3)−iv),
$k=0,$ for 4).

Hence we can choose an orthonormal basis $\{f_k(\omega)\}$ in $\mathfrak{H}(\omega)$, such that $f_k(\omega)$ is contained in $H_k(\omega)$, these vectors are called weight vectors. Here we select it as in the Bargmann's paper [1].

For brevity, we denote $f_k(D_m^+)$ by f_k^m and $\mathfrak{H}(D_m^+)$ by $\mathfrak{H}_m$.

2. The knowledges on decompositions of Kronecker products of these representations are not complete. But we can claim the following lemmas.

Lemma 1. *$D_m^+ \otimes D_n^+$ is decomposed to the discrete direct sum of irreducible components as*

$$D_m^+ \otimes D_n^+ \cong \sum_p \oplus D_p^+ \quad (p \geqq m+n,\ p+m+n;\ integer).$$

And let this isomorphism be φ, then $\langle \varphi(f_k^m \otimes f_j^n), f_h^p \rangle$ is zero for $h \neq k+j$, and not zero when $h=k+j$ for any admissible k, j, p. Especially for $D_{1/2}^+ \otimes D_{1/2}^+$, f_p^p in the component D_p^+ is expanded as

$$f_p^p = \sum_{s=0}^{p-1} a_{p,s} \varphi(f_{s+1/2}^{1/2} \otimes f_{p-s-1/2}^{1/2}),$$

and

$$a_{p,0} = a_{p,1} \neq 0.$$

Lemma 2. *$D_m^+ \otimes \omega$ ($\omega = C_l^t$ or D_n^-) contains D_p^+ ($p>1/2$ or $m-n \geqq p > 1/2$, and $p+m+t$ or $p+m+n$; integer) as a discrete component with multiplicity one and put*

$$\langle \varphi(f_n^m \otimes f_k(\omega)), f_p^p \rangle = \delta_{p,n-k} \cdot c(\omega, k, m, p),$$

then $c(\omega, k, m, p) \neq 0$, for any admissible k, p.

These lemmas are immediately assured by calculations of weight vectors as in Pukánszky's work (cf. [2] chap. II). So we don't repeat here.

3. Now we prove the theorem. At first, we call a normalized vector v in $\mathfrak{H}_{1/2}$ *fundamental* if $\varphi(v \otimes v)$ belongs to $\mathfrak{H}_1$ in the irreducible decomposition. We consider the condition that

$$v = \sum_{k \geqq 0} a_k f_{k-1/2}^{1/2} \tag{1}$$

is fundamental.

It is easy to see that for any unitary admissible operator field $T=\{T(\omega)\}$, the vector $T(D_{1/2}^+) f_{1/2}^{1/2}$ is fundamental. Especially for fixed g in G, $U_g(D_{1/2}^+) f_{1/2}^{1/2}$ is fundamental.

If v is fundamental then direct calculation shows

$$c(s) f_s^{1/2} = a_k a_{s-k-1} \varphi(f_{k+1/2}^{1/2} \otimes f_{s-k-1/2}^{1/2}),$$

where $c(s)$ is a constant depending only to s. From Lemma 1, and linear independency of the family $\{f_{k+1/2}^{1/2} \otimes f_{s-k-1/2}^{1/2}\}$, we get

$$c(s) a_{s,k} = a_k a_{s-k-1},$$

i.e. $$a_{s,0}a_1a_{s-2}=a_{s,1}a_0a_{s-1}.$$
Considering $a_{s,0}=a_{s,1}\neq 0$ in Lemma 1, we get
$$a_s=(a_1/a_0)^s a_0.$$
And condition for the convergence of the series in (1) is
$$|a_1/a_0|<1, \tag{2}$$
the normalizing condition is
$$|a_0|=(1-|a_1/a_0|^2)^{1/2}. \tag{3}$$
That is, fundamental vector is decided by the value, $a_0=\langle v, f_{1/2}^{1/2}\rangle$ and $a_1/a_0=\langle v, f_{3/2}^{1/2}\rangle/\langle v, f_{1/2}^{1/2}\rangle$, satisfying the conditions (2) and (3).

In the other hand, if we take the unique expression
$$g=\begin{pmatrix}\cos\theta/2, & -\sin\theta/2\\ \sin\theta/2, & \cos\theta/2\end{pmatrix}\begin{pmatrix}\mathrm{ch}\,t/2, & \mathrm{sh}\,t/2\\ \mathrm{sh}\,t/2, & \mathrm{ch}\,t/2\end{pmatrix}\begin{pmatrix}\cos\phi/2, & -\sin\phi/2\\ \sin\phi/2, & \cos\phi/2\end{pmatrix},$$
$$-2\pi<\theta\leqq 2\pi,\ 0\leqq t<\infty,\ -\pi<\phi\leqq\pi,$$
then the matrix elements are
$$\langle U_g(D_{1/2}^+)f_{1/2}^{1/2}, f_{1/2}^{1/2}\rangle=\exp(i(\theta+\phi)/2)(\mathrm{ch}\,t/2)^{-1},$$
$$\langle U_g(D_{1/2}^+)f_{1/2}^{1/2}, f_{3/2}^{1/2}\rangle/\langle U_g(D_{1/2}^+)f_{1/2}^{1/2}, f_{1/2}^{1/2}\rangle=c_1\exp(i\theta)\tanh t/2,\quad |c_1|=1.$$
So the following lemma is valid.

Lemma 3. *For any fundamental vector v, an element g in G is uniquely determined and*
$$v=U_g(D_{1/2}^+)f_{1/2}^{1/2}.$$
Particularly,

Corollary. *For any unitary admissible operator field $\boldsymbol{T}=\{T(\omega)\}$ there is unique element g in G such that*
$$U_g(D_{1/2}^+)f_{1/2}^{1/2}=T(D_{1/2}^+)f_{1/2}^{1/2}.$$

Moreover the classical theory on non-euclidean space G/R shows that the topology of G coincides with the weak topology of totality of fundamental vectors in $\mathfrak{H}_{1/2}$ by this correspondences.

Now we show that each unitary admissible operator field $\{T(\omega)\}$ is uniquely determined by the vector $T(D_{1/2}^+)f_{1/2}^{1/2}$. This assertion leads us to prove the main theorem immediately.

I) At first the vector $T(D_m^+)f_m^m$ is determined by the recurrence formula with respect to m,
$$\begin{aligned}T(D_m^+)f_m^m&=T(D_m^+)(\varphi(c_m\cdot f_{1/2}^{1/2}\otimes f_{m-1/2}^{m-1/2}))\\&=c_m\varphi(T(D_{1/2}^+)f_{1/2}^{1/2}\otimes T(D_{m-1/2}^+)f_{m-1/2}^{m-1/2}).\end{aligned}$$

From Lemma 1, the vector $T(D_m^+)f_{m+k-1}^m$ is characterized as a vector v in $\mathfrak{H}_m$ such that

i) $$T(D_{1/2}^+)f_{1/2}^{1/2}\otimes v\in\sum_{k\geqq s\geqq 1}\otimes\mathfrak{H}_{m+s-1/2}.$$

ii) $$v\perp T(D_m^+)f_{m+s-1}^m,\quad\text{for}\quad k>s\geqq 1.$$

iii) $$\begin{aligned}&\langle\varphi(T(D_{1/2}^+)f_{1/2}^{1/2}\otimes v),\ T(D_{m+k-1/2}^+)f_{m+k-1/2}^{m+k-1/2}\rangle\\&=\langle\varphi(f_{1/2}^{1/2}\otimes f_{m+k-1}^m),\ f_{m+k-1/2}^{m+k-1/2}\rangle\quad(\neq 0).\end{aligned}$$

That is, $T(D_m^+)$ is determined from $T(D_{1/2}^+)f_{1/2}^{1/2}$.

II) Lastly from Lemma 2, if ω is C_l^t or D_n^-, then for any positive integer s such that $s>1/2$ or $m-n\geqq s>1/2$, and $s+m+t$ or $s+m+n$ is an integer, $D_m^+\otimes\omega$ contains D_s^+ as a discrete irreducible component.

And for given v, in $\mathfrak{H}(\omega)$, let

$$v=\sum_k b_k f_k(\omega),$$

then from Lemma 2,

$$\langle f_n^m\otimes v, f_s^s\rangle=\sum_k b_k\langle f_n^m\otimes f_k(\omega), f_s^s\rangle=b_k c(\omega, m, n, s),$$

i.e.

$$b_k=(c(\omega, m, n, s))^{-1}\langle f_n^m(D_m^+)\otimes v, f_s^s\rangle.$$

This means that the vector v is determined if the coefficient $\langle f_n^m\otimes v, f_s^s\rangle$ is given. Now in this equation, let $v=T(\omega)v_0$ for given v_0, then the equality

$$\langle f_n^m\otimes T(\omega)v_0, f_s^s\rangle=\langle\varphi(T(D_m^+))^{-1}f_n^m\otimes v_0), (T(D_s^+))^{-1}f_s^s\rangle,$$

gives the vector $T(\omega)v_0$ from v_0, $(T(D_m^+))^{-1}f_n^m$, and $(T(D_s^+))^{-1}f_s^s$, therefore, using the results of I), from $T(D_{1/2}^+)f_{1/2}^{1/2}$. This completes the proof.

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References

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