134. The Number of Irreducible Components of an Ideal and the Semi-Regularity of a Local Ring

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Let Q be a local ring with the maximal ideal $\mathfrak{m}$, and $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of Q. Then it is known that the number n of irreducible components of $\mathfrak{q}$ is equal to the length $L_Q(\mathfrak{q}:\mathfrak{m}/\mathfrak{q})$ of Q-module $\mathfrak{q}:\mathfrak{m}/\mathfrak{q}$, which was defined as the index of reducibility of $\mathfrak{q}$ by D. G. Northcott in his paper [1]. In the same paper, he proved that, *if Q is semi-regular, then the index of reducibility of an $\mathfrak{m}$-primary ideal generated by a system of parameters depends only on Q, and not on the choice of the system of parameters* (Theorem 3, [1]). Gröbner's theorem, i.e. *in a regular local ring, any ideal which can be generated by a system of parameters is irreducible,* is a special case of this theorem.

On the other hand, it is known that, in a local ring Q, *if the index of reducibility of an $\mathfrak{m}$-primary ideal generated by a system of parameters is equal to* 1 *constantly* (i.e. *if every ideal generated by a system of parameters is irreducible*), *then the ring Q is semi-regular* (see [2]). However it should be noticed that the condition of this proposition is not a sufficient condition for the regularity of Q.

Concerning these results, the following question may be raised:

Can we conclude that the local ring Q is semi-regular if the index of reducibility of an $\mathfrak{m}$-primary ideal generated by a system of parameters is equal to some constant which is not necessarily 1?

The main purpose of this paper is to answer this question.

Now we shall begin by proving the following theorem:

Theorem 1. *Let Q be a local ring of dimension d, and $\mathfrak{m}$ be its maximal ideal. If $\mathfrak{m}$ is generated by d or $d+1$ elements, and if the index of reducibility of an $\mathfrak{m}$-primary ideal generated by a system of parameters is equal to some constant which does not depend on the choice of such an ideal, then every $\mathfrak{m}$-primary ideal generated by a system of parameters is irreducible.*

Proof. If $\mathfrak{m}$ is generated by d elements, then Q is regular, and the conclusion follows (see Gröbner [3] or Northcott [1]).

Now we shall assume that the minimal basis of $\mathfrak{m}$ consists of $d+1$ elements. Then it is easy to see that we can assume that $\mathfrak{m}$ is generated by $u_1, u_2, \cdots, u_d, u_{d+1}$ where $\{u_1, u_2, \cdots, u_d\}$ is a system of

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parameters. Consider $L_Q\left(\left(\sum_{i=1}^{d} Qu_i\right) : \mathfrak{m} \middle/ \sum_{i=1}^{d} Qu_i\right)$. We shall prove that it is equal to 1. Let $\bar{Q}$ be $Q\middle/\sum_{i=1}^{d} Qu_i$, and $\bar{\mathfrak{m}}$ be $\mathfrak{m}\middle/\sum_{i=1}^{d} Qu_i$. Then $\bar{Q}$ is zero-dimensional, and $\bar{\mathfrak{m}}$ is generated by an element $\bar{u}_{d+1}$. Let t be an integer such that $\bar{u}_{d+1}^t=0$, $\bar{u}_{d+1}^{t-1}\neq 0$, then $(0):\bar{\mathfrak{m}}=(\bar{u}_{d+1}^{t-1})$. Consequently $L_{\bar{Q}}((0):\bar{\mathfrak{m}})=1$. Therefore $L_Q\left(\left(\sum_{i=1}^{d} Qu_i\right) : \mathfrak{m} \middle/ \sum_{i=1}^{d} Qu_i\right)=1$. Hence, by the hypothesis of this theorem, the index of reducibility of an ideal generated by a system of parameters is equal to 1 (i.e. such an ideal is irreducible). Thus we have proved the theorem.

Corollary 1. *If Q is a local ring satisfying the conditions of Theorem 1, then Q is semi-regular.*

Proof. It is known that, if every $\mathfrak{m}$-primary ideal generated by a system of parameters is irreducible, then the ring is semi-regular (see [2]). Therefore Corollary 1 follows from the Theorem.

A local ring, in which every ideal generated by a system of parameters is irreducible, is called Gorenstein ring. Now we shall add another corollary to be used later.

Corollary 2. *Let Q be a d-dimensional semi-regular local ring, and let $\mathfrak{m}$ be its maximal ideal. If $\mathfrak{m}$ is generated by d or $d+1$ elements, then Q is a Gorenstein ring.* (This corollary, for the case $d=1$, was given in [4], Prop. 6.4.)

Proof. By virtue of the result due to Northcott [1], Corollary 2 is a direct consequence of Theorem 1.

Theorem 1 gives an affirmative answer to the question stated at the beginning of this paper in a special case. However, we shall show that, if the dimension of Q is d and its maximal ideal is generated by $d'(d'\geqq d+2)$ elements, then the answer to the question is negative. We shall give a counter example.

Let k be a field, $F=k[[X, Y, Z_1, Z_2,\cdots, Z_n]]$ be a formal power series ring in $n+2$ indeterminates $X, Y, Z_1, Z_2,\cdots, Z_n$, and $\mathfrak{M}$ be the maximal ideal of F. Suppose ξ be an element of F such that $\xi\in\mathfrak{M}^2$, $\xi\notin\sum_{i=1}^{n} FZ_i$ and the image of ξ in the residue ring $F\middle/\sum_{i=1}^{n} FZ_i$ is prime. (If k does not contain $\sqrt{-1}$, X^2+Y^2 may be the simplest example of such ξ. If k contains $\sqrt{-1}$, X^2-Y^3 may be the simplest one.) It is obvious that the residue ring $Q=F\middle/\left(F\xi+\sum_{i=1}^{n}\mathfrak{M}Z_i\right)$ is a one-dimensional local ring and its maximal ideal $\mathfrak{m}$ is generated by $x, y, z_1, z_2,\cdots, z_n$ which are the residue classes of $X, Y, Z_1, Z_2,\cdots, Z_n$ respectively. $\{x, y, z_1, z_2,\cdots, z_n\}$ is a minimal basis of $\mathfrak{m}$, since $\xi\in\mathfrak{M}^2$. Now we shall prove:

Theorem 2. *Let Q be a ring defined above. Then*

(i) *Q is not semi-regular,*

(ii) $L_Q(Qu:\mathfrak{m}/Qu)=n+1$, *where* u $(u\in\mathfrak{m})$ *is a parameter.*

Proof. by the definition of Q, we have $\mathfrak{m}z_i=(0)$, $i=1, 2, \cdots, n$, consequently $\mathfrak{m}$ consists of zero divisors only. So (i) follows immediately.

In order to prove (ii), we need the following three lemmas:

Lemma 1.

(i) *u is a parameter in Q if and only if* $u\notin\sum_{i=1}^{n}Qz_i$.

(ii) $Qu+\sum_{i=1}^{t}Qz_i\supsetneqq Qu+\sum_{i=1}^{t-1}Qz_i$, $t=1, 2,\cdots, n$.

Proof. It is easy to see that $\sum_{i=1}^{n}Qz_i$ is a prime ideal belonging to (0), and is a nilradical. (i) is a direct consequence of this fact.

We shall first prove the strict inclusion $Qu+Qz_1\supsetneqq Qu$. Assuming that $Qu+Qz_1=Qu$, we shall obtain a contradiction. From the assumption, it follows that $z_1\in Qu\frown Qz_1\subset Qu\frown\sum_{i=1}^{n}Qz_i$. Since $\sum_{i=1}^{n}Qz_i$ is prime, $Qu\frown\sum_{i=1}^{n}Qz_i=\left(\sum_{i=1}^{n}Qz_i\right)u\subset\sum_{i=1}^{n}\mathfrak{m}z_i=(0)$. On the other hand, as stated above, z_1 is an element of minimal basis of $\mathfrak{m}$, and is not equal to zero, which is a contradiction.

Hence follows the strict inclusion $Qu+\sum_{i=1}^{t}Qz_i\supsetneqq Qu+\sum_{i=1}^{t-1}Qz_i$. To prove this, we have only to consider the residue ring $Q\Big/\left(\sum_{i=1}^{t-1}Qz_i\right)$.

Lemma 2.

$$L_Q\left(\left(Qu+\sum_{i=1}^{n}Qz_i\right)\ Qu\right)=n.$$

Proof. Since $\mathfrak{m}z_t=(0)$, $L_Q\left(\left(Qu+\sum_{i=1}^{t}Qz_i\right)\Big/\left(Qu+\sum_{i=1}^{t-1}Qz_i\right)\right)=1$ by Lemma 1. Lemma 2 follows immediately.

Lemma 3. *Let u be a parameter in Q, then*

$$\left(Qu+\sum_{i=1}^{n}Qz_i\right):\mathfrak{m}=\left(Qu+\sum_{i=1}^{n-1}Qz_i\right):\mathfrak{m}=\left(Qu+\sum_{i=1}^{n-2}Qz_i\right):\mathfrak{m}=\cdots$$
$$\cdots=(Qu+Qz_1):\mathfrak{m}=Qu:\mathfrak{m}.$$

Proof. First we shall prove the equality

$$(Qu+Qz_1):\mathfrak{m}=Qu:\mathfrak{m}. \tag{1}$$

Since the inclusion $(Qu+Qz_1):\mathfrak{m}\supset Qu:\mathfrak{m}$ is obvious, we have only to prove the inverse inclusion. Let t be an element of $(Qu+Qz_1):\mathfrak{m}$. We have $xt=ru+sz_1$, $yt=r'u+s'z_1$ for some $r, s, r', s'\in Q$. From these two equations, it follows that $(r'x-ry)t=(r's-rs')z_1$. If t belongs to $\sum_{i=1}^{n}Qz_i$, then $t\in(0):\mathfrak{m}\subset Qu:\mathfrak{m}$. Then we have nothing to prove. Therefore we shall assume that $t\notin\sum_{i=1}^{n}Qz_i$. Since $\sum_{i=1}^{n}Qz_i$ is prime, $(r'x-ry)\in Qz_1:t\subset\left(\sum_{i=1}^{n}Qz_i\right):t=\sum_{i=1}^{n}Qz_i$. Since $\{x, y, z_1, z_2,\cdots, z_n\}$ is a minimal basis of $\mathfrak{m}$, none of r and r' can be a unit element. Since $xt, yt \in \mathfrak{m}^2$ and $ru, r'u \in \mathfrak{m}^2$, it follows that $sz_1, s'z_1 \in \mathfrak{m}^2$, consequently $s, s' \in \mathfrak{m}$. Therefore $xt = ru$, $yt = r'u$ since $\mathfrak{m}z_1 = (0)$. Hence we have $\mathfrak{m}t = \left(Qx + Qy + \sum_{i=1}^{n} Qz_i\right)t = (Qx + Qy)t \subset Qu$, consequently t belongs to $Qu : \mathfrak{m}$. Thus we have proved the equality (1).

We can prove the equality $\left(Qu + \sum_{i=1}^{t} Qz_i\right) : \mathfrak{m} = \left(Qu + \sum_{i=1}^{t-1} Qz_i\right) : \mathfrak{m}$ by considering the residue ring $Q\Big/\left(\sum_{i=1}^{t-1} Qz_i\right)$. q. e. d.

Proof of Theorem 2 (*continued*). By virtue of these lemmas, we have

$$\begin{aligned} &L_Q((Qu : \mathfrak{m})/Qu) \\ &= L_Q\left(\left(Qu + \sum_{i=1}^{n} Qz_i\right) : \mathfrak{m}\Big/\left(Qu + \sum_{i=1}^{n} Qz_i\right)\right) + L_Q\left(\left(Qu + \sum_{i=1}^{n} Qz_i\right)\Big/Qu\right) \\ &= L_Q\left(\left(Qu + \sum_{i=1}^{n} Qz_i\right) : \mathfrak{m}\Big/\left(Qu + \sum_{i=1}^{n} Qz_i\right)\right) + n. \end{aligned}$$

Let $\bar{Q}$ be the residue ring $Q/\sum_{i=1}^{n} Qz_i$, $\bar{\mathfrak{m}}$ be $\mathfrak{m}/\sum_{i=1}^{n} Qz_i$. It is easy to see that $\bar{Q}$ is a one-dimensional semi-regular local ring and that the maximal ideal $\bar{\mathfrak{m}}$ is generated by two elements. Then, by Corollary 2 to Theorem 1, the index of reducibility of an ideal generated by a parameter in $\bar{Q}$ such as $\bar{Q}\bar{u} = \left(Qu + \sum_{i=1}^{n} Qz_i\right)\Big/\left(\sum_{i=1}^{n} Qz_i\right)$ is equal to 1. Therefore

$$L_Q\left(Qu + \sum_{i=1}^{n} Qz_i\right) : \mathfrak{m}\Big/\left(Qu + \sum_{i=1}^{n} Qz_i\right)\Big) = L_{\bar{Q}}(\bar{Q}\bar{u} : \bar{\mathfrak{m}}/\bar{Q}\bar{u}) = 1.$$

This completes the proof of Theorem 2.

References

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