155. On Closures of Vector Subspaces. I

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1. Let E be a vector space, and let M be an infinite dimensional vector subspace of E. In a previous paper,[1] we stated a condition which ensures the existence of a locally convex metrizable topology on E possessing the following properties:

(1) M is dense in E;

(2) The induced topology on M is finer than a given locally convex metrizable topology on M.

Moreover, as a consequence of it, we obtained a condition which ensures the existence of a locally convex metrizable topology on E satisfying merely the requirement (1). The main interest in the present paper is on the requirement (1), and we shall concern, in what follows, with the problem of existence of a locally convex Hausdorff topology on E possessing the property (1) without the restriction that the topology is metrizable.

The terminology and notations used in the previous paper will be continued in this paper.

2. Throughout this section the operation of polar will be taken in the dual system (E^{**}, E^{*}).

We have immediately the following lemmas.

LEMMA 1. *Let E be a vector space, and let E' be a vector subspace of E^{*}. If the dual system (E, E') is separated, then E'° is a $\sigma(E^{**}, E^{*})$-closed vector subspace of E^{**} contained in an algebraic supplement of E in E^{**}. Conversely, if F is a $\sigma(E^{**}, E^{*})$-closed vector subspace of E^{**} such that $E \frown F=\{0\}$, then the dual system (E, F°) is separated.*

LEMMA 2. *Let E be a locally convex vector space, and let E' be its dual. For every vector subspace M of E, the following conditions are equivalent:*

(1) M is dense in E.

(2) $M^{\circ} \frown E'=\{0\}$.

(3) The vector subspace $M^{\circ\circ}+E'^{\circ}$ of E^{**} is dense in E^{**} for the weak topology $\sigma(E^{**}, E^{*})$.

Thus we have

THEOREM 1. *Let M be a vector subspace of a vector space E.*

1) S. Kasahara: Locally convex metrizable topologies which make a given vector subspace dense. Proc. Japan Acad., **40**, 718-722 (1964).

Then there exists a locally convex Hausdorff topology on E *for which* M *is dense in* E *if and only if there exists a* $\sigma(E^{**}, E^{*})$*-closed vector subspace* F *of* E^{**} *such that* $E \frown F=\{0\}$ *and* $M^{\circ\circ}+F$ *is dense in* E^{**} *for the weak topology* $\sigma(E^{**}, E^{*})$.

Moreover, it follows from the above lemmas the following

THEOREM 2. *Let* E *be a locally convex Hausdorff vector space, and let* M *be a vector subspace of* E. *If* M *is dense in* E, *then every algebraic supplement of* M *in* E *is not dense in* E.

Proof. Denote by E' the dual of E, and suppose that both M and an algebraic supplement N of M in E are dense in E. Then by Lemma 2, we have $(M^{\circ} \smile N^{\circ}) \frown E'=\{0\}$, and hence $(M^{\circ\circ} \frown N^{\circ\circ})+E'^{\circ}$ is dense in E^{**} for the weak topology $\sigma(E^{**}, E^{*})$. But since $M^{\circ\circ} \frown N^{\circ\circ} =\{0\}$, this means that E'° is dense in E^{**}, which contradicts, in view of Lemma 1, the fact that the dual system (E, E') is separated.

COROLLARY. *Let* E *be a locally convex Hausdorff vector space, and let* M, N *be two vector subspaces of* E *with the intersection* $\{0\}$.

1° *If* M *is dense in* E, *then* N *is not dense in* E.

2° *If* N *is contained in the closure of* M, *then* M *is not contained in the closure of* N.

THEOREM 3. *Let* M *be a vector subspace of a vector space* E, *and let* τ_0 *be a locally convex Hausdorff topology on* M. *Then, for every algebraic supplement* N *of* M, *there exists a locally convex Hausdorff topology* τ *on* E *such that both* M *and* N *are closed, and the induced topology of* τ *on* M *coincides with* τ_0. *Moreover, if* τ_0 *is metrizable, then* τ *can be chosen to be metrizable.*

Proof. We introduce a norm topology to the vector subspace N. Since each $x \in E$ can be written uniquely in the form: $x=y+z$, $y \in M$, $z \in N$, by setting, then, $p(x)=y$ a linear mapping p of E onto M is obtained. Let e be the identity mapping of E onto itself. Then it is easy to see that the weakest topology on E which makes both mappings p and $e-p$ continuous possesses the required property.

3. In this section we give another characterization of separateness of a dual system.

Let (E, E') be a dual system, and let A' be a subset of E' containing a base of E'. We denote by $\mathcal{F}(A')$ the vector space consisting of all complex-valued functions on A'. It is straightforward to see that the mapping of E into the vector space $\mathcal{F}(A')$, which maps $x \in E$ to the function f_x defined by

$$f_x(x')=\langle x, x'\rangle \qquad \text{for all } x' \in A',$$

is linear. The following Lemma can be proved easily.

LEMMA 3. *The following conditions on a dual system* (E, E') *are equivalent:*

(1) (E, E') *is separated in* E.

(2) *For every subset* A' *of* E' *containing a base of* E', *the mapping* $x \to f_x$ *of* E *into the vector space* $\mathcal{F}(A')$ *is an isomorphism.*

(3) *For every base* B_E *of* E, *and for every subset* A' *of* E' *containing a base of* E', *the mapping* $x \to f_x$ *of* B_E *into* $\mathcal{F}(A')$ *is one-to-one, and the subset* $\{f_x;\ x \in B_E\}$ *of* $\mathcal{F}(A')$ *is linearly independent.*

Let E be an infinite dimensional vector space, and let E' be a vector subspace of E^*. It is known[2] that the dimension of E^* is equal to $2^{\dim(E)}$. Consequently, we have $\dim(E) \leqq \dim(E'^*) = 2^{\dim(E')}$, because E can be identified with a vector subspace of the algebraic dual E'^* of E'. Conversely we have

LEMMA 4. *Let* E *be an infinite dimensional vector space. If* $\dim(E) \leqq 2^{\mathfrak{a}}$ *for a cardinal number* $\mathfrak{a}$, *then there exists a vector subspace* $E' \subseteq E^*$ *of dimension* $\leqq \mathfrak{a}$ *such that the dual system* (E, E') *is separated.*

Proof. Let A be a set with the cardinal number $\mathfrak{a}$, and consider the vector space $\mathcal{F}(A)$. Let B_E be a base of E, and let $\boldsymbol{B}$ be a base of $\mathcal{F}(A)$. Then it follows from the inequality $\dim(E) \leqq 2^{\mathfrak{a}} = \dim(\mathcal{F}(A))$[2] that there exists a one-to-one mapping $x \to f_x$ of B_E into $\boldsymbol{B}$. For each $\alpha \in A$, we define a linear functional α' on E by setting

$$\langle x, \alpha' \rangle = f_x(\alpha) \qquad \text{for all } x \in B_E.$$

Then it is evident that the vector subspace $E' \subseteq E^*$ spanned by the set $\{\alpha';\ \alpha \in A\}$ is of dimension $\leqq \mathfrak{a}$. The separateness of the dual system (E, E') follows from Lemma 3.

4. We shall prove now the following

THEOREM 4. *Let* M *be a vector subspace of an infinite dimensional vector space* E, *and let* τ_0 *be a locally convex Hausdorff topology on* M. *Then there exists a locally convex Hausdorff topology* τ *on* E *which makes* M *dense in* E *and induces on* M *a topology coarser than* τ_0 *if and only if* $\dim(E) \leqq 2^{\dim(M')}$, *where* M' *is the dual of* M *for* τ_0.

Proof of the "only if" part. For each element x' of the dual E' of E for the topology τ, its restriction $x'|_M$ to M does belong to M'. Since $x'|_M = y'|_M$ implies $x' - y' \in M^\circ \frown E' = \{0\}$, the mapping $x' \to x'|_M$ of E' into M' is one-to-one. In addition, as can be readily seen, this mapping is linear. Consequently we have $\dim(E') \leqq \dim(M')$. On the other hand, because the dual system (E, E') is separated, we have $\dim(E) \leqq 2^{\dim(E')}$, and hence we have $\dim(E) \leqq 2^{\dim(M')}$.

Proof of the "if" part. We denote by B_M a base of M, and by $B_{M'}$ a base of M'. For each $x \in B_M$, put

$$f_x(x') = \langle x, x' \rangle \qquad \text{for all } x' \in B_{M'}.$$

Then since the dual system (M, M') is separated, the mapping $x \to f_x$ of B_M into the vector space $\mathcal{F}(B_{M'})$ is one-to-one, and the set $\boldsymbol{B}_M$ of all f_x, $x \in B_M$, is linearly independent (see Lemma 3). Therefore, the

2) See G. Köthe: Topologische lineare Räume, I. Springer-Verlag, Berlin (1960).

cardinal number of the set $\boldsymbol{B}_M$ is equal to the dimension of M. Let B_N be a base of an algebraic supplement N of M in E. Then by the assumption, we have $\dim(M)+\dim(N)\leqq 2^{\dim(M')}=\dim(\mathcal{F}(B_{M'}))$, and hence we can find a linearly independent subset $\boldsymbol{B}\subseteq\mathcal{F}(B_{M'})$ with cardinal number $\dim(N)$ such that $\boldsymbol{B}\frown\boldsymbol{B}_M=\phi$. Take a one-to-one mapping $x\to\varphi_x$ of B_N onto $\boldsymbol{B}$, and define, for each $x'\in B_{M'}$, a linear functional $\bar{x}'$ on E by letting

$$\langle x, \bar{x}'\rangle=\begin{cases}\langle x, x'\rangle & \text{for } x\in M,\\ \varphi_x(x') & \text{for } x\in B_N.\end{cases}$$

Then the space E and the vector subspace E' of E^* spanned by the set $A'=\{\bar{x}'; x'\in B_{M'}\}$ forms a separated dual system (E, E'). To prove this, consider, for each $x\in B_M\smile B_N$, a complex-valued function g_x on A' defined by

$$g_x(\bar{x}')=\langle x, \bar{x}'\rangle \qquad \text{for all } \bar{x}'\in A'.$$

Since for every $x'\in B_{M'}$

$$g_x(\bar{x}')=\begin{cases}f_x(x') & \text{whenever } x\in B_M,\\ \varphi_x(x') & \text{whenever } x\in B_N,\end{cases}$$

it follows that the mapping $x\to g_x$ of $B_M\smile B_N$ into $\mathcal{F}(A')$ is one-to-one. We shall show that the image of the set $B_M\smile B_N$ under this mapping is a linearly independent subset of the vector space $\mathcal{F}(A')$. Suppose that $\sum_{i=1}^k \lambda_i g_{x_i}=0$, where $x_1,\cdots,x_m\in B_M$ and $x_{m+1},\cdots,x_k\in B_N$. Then, for every $\bar{x}'\in A'$ we have

$$0=\sum_{i=1}^k \lambda_i g_{x_i}(\bar{x}')=\sum_{i=1}^m \lambda_i\langle x_i, x'\rangle+\sum_{i=m+1}^k \lambda_i\varphi_{x_i}(x')$$
$$=\sum_{i=1}^m \lambda_i f_{x_i}(x')+\sum_{i=m+1}^k \lambda_i\varphi_{x_i}(x').$$

This shows that $\sum_{i=1}^m \lambda_i f_{x_i}+\sum_{i=m+1}^k \lambda_i\varphi_{x_i}=0$ in the vector space $\mathcal{F}(B_{M'})$, and so we have $\lambda_i=0$ for $i=1,\cdots,k$. Hence the image of the set $B_M\smile B_N$ under the mapping $x\to g_x$ is linearly independent. Thus according to Lemma 3, the dual system (E, E') is separated. Therefore the weak topology $\sigma(E, E')$ is Hausdorff one.

Now we shall proceed to prove that the vector subspace M is dense in E for the weak topology $\sigma(E, E')$. Let $x'=\sum_{i=1}^n \lambda_i\bar{x}'_i$, $x'_i\in B_{M'}$, be an element of $M^\circ\frown E'$. Then we have

$$\langle x, \sum_{i=1}^n \lambda_i x'_i\rangle=\sum_{i=1}^n \lambda_i\langle x, x'_i\rangle=\sum_{i=1}^n \lambda_i\langle x, \bar{x}'_i\rangle$$
$$=\langle x, x'\rangle=0 \qquad \text{for all } x\in M.$$

It follows that $\sum_{i=1}^n \lambda_i x'_i=0$ in M', and hence we have $\lambda_i=0$ for $i=1,\cdots,n$. Consequently we have $M^\circ\frown E'=\{0\}$, which shows that M is dense in E for the weak topology $\sigma(E, E')$.

Let x' be a non-zero element of E'; then its restriction $x'|_M$ to

M belongs to M', because so does each element of A'. Hence, for any $x'_1, \cdots, x'_n \in E'$, we can find a neighborhood U of 0 in M for the topology τ_0 such that

$$|\langle x, x'_i \rangle| = |\langle x, x'_i|_M \rangle| \leqq 1, \qquad i=1, \cdots, n,$$

for every $x \in U$. Therefore U is contained in $(\{x'_1, \cdots, x'_n\})^\circ \frown M$ and consequently the induced topology of $\sigma(E, E')$ on M is coarser than τ_0. This completes the proof of the "if" part.

As a consequence of Theorem 4, we have the following

THEOREM 5. *Let M be a vector subspace of an infinite dimensional vector space E. Then there exists a locally convex Hausdorff topology on E for which M is dense in E if and only if* $\dim(E) \leqq 2^{\mathfrak{a}}$, *where* $\mathfrak{a} = 2^{\dim(M)}$.

Proof. Assume $\dim(E) \leqq 2^{\mathfrak{a}}$. Since the dual system (M, M^*) is separated, the weak topology $\sigma(M, M^*)$ is a locally convex Hausdorff topology on M. Moreover, from the fact that $\dim(M^*) = 2^{\dim(M)}$, it follows that $\dim(E) \leqq 2^{\dim(M^*)}$. Thus the conclusion follows from Theorem 4. Conversely suppose that M is dense in E for a locally convex Hausdorff topology τ on E. Then, by applying Theorem 4 to the induced topology τ_0 of τ on M, we have $\dim(E) \leqq 2^{\dim(M')}$. But then, since $\dim(M') \leqq \dim(M^*) = 2^{\dim(M)}$, we have $\dim(E) \leqq 2^{\mathfrak{a}}$.

Combining these theorems with Theorem 3, we have the following

COROLLARY 1. *Let E be a vector space, and let M, N be two vector subspaces of E. Suppose that the dimension of M is infinite, and M is occupied by a locally convex Hausdorff topology τ_0. Then there exists a locally convex Hausdorff topology τ on E such that the closure of M contains N and τ induces on M a topology coarser than τ_0 if and only if* $\dim(M+N) \leqq 2^{\dim(M')}$, *where M' is the dual of M.*

COROLLARY 2. *Let E be a vector space, and let M, N be two vector subspaces of E. Suppose that the dimension of M is infinite. Then there exists a locally convex Hausdorff topology on E such that the closure of M contains N if and only if* $\dim(M+N) \leqq 2^{\mathfrak{a}}$, *where* $\mathfrak{a} = 2^{\dim(M)}$.

(This article is dedicated to Professor K. Kunugi in celebration of his 60th birthday.)