# 15. *Some Remarks on Von Neumann Algebras with an Algebraical Property*

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(Comm. by Kinjirô KUNUGI, M.J.A., Jan. 12, 1965)

Schwartz [8] established that there exists a pair of non-isomorphic, non-hyperfinite finite factors, so that he introduced the property $P$ which is spatial one. In [2], we introduced a purely algebraical property, the property $Q$, and showed that the results of Schwartz were followed by the property $Q$. In [3], we proved that the crossed product $G\otimes\mathcal{A}$ of the finite von Neumann algebra $\mathcal{A}$ by a group $G$ of outer automorphisms of $\mathcal{A}$ has the property $Q$ only if $G$ is amenable, and that the factor constructed by an enumerable ergodic $m$-group $G$ on a measure space by the method due to Murray-von Neumann [4] is a continuous hyperfinite factor only if $G$ is amenable. We obtained also in [3] a sufficient condition for the crossed product to have the property $P$.

In this paper, we shall report some further results on von Neumann algebras with the property $Q$. We shall publish the details in the Memoir of Osaka Gakugei University, Sect. B, No. 13 (1964).

In the below, we shall use the terminology of [3] without further explanations.

In the first place, a sufficient condition for the crossed product to have the property $Q$ will be given in the following theorem:

THEOREM 1. *Let $\mathcal{A}$ be a von Neumann algebra with a finite faithful normal trace $\varphi$ and $G$ an amenable group of outer automorphisms of $\mathcal{A}$ such that*

$$\varphi(A^g)=\varphi(A) \qquad \text{for } g\in G \text{ and } A\in\mathcal{A}.$$

*If $\mathcal{A}$ has an amenable generator $\mathcal{G}$ satisfying the following conditions:*

i) $$\mathcal{G}^g\subset\mathcal{G} \qquad \text{for any } g\in G,$$

*and*

ii) $$\int f(U)\,dU^g=\int f(U)\,dU \qquad \text{for } g\in G \text{ and } f\in L^\infty(\mathcal{G}),$$

*then the crossed product $G\otimes\mathcal{A}$ has the property $Q$.*

Let $\Phi$ be the free group with two generators, then there exists a group $G$ of outer automorphisms of the hyperfinite continuous factor $\mathcal{A}$ which is isomorphic to $\Phi$, cf.[6]. For an infinite dimensional Hilbert space $\mathcal{K}$, let $\mathcal{B}=\mathcal{L}(\mathcal{K})$, then $(G\otimes\mathcal{A})\otimes\mathcal{B}$ is a factor of type $\mathrm{II}_\infty$ without the property $Q$. Hence, we have the following

THEOREM 2. *There is a factor of type $II_\infty$ which has not the property Q.*

In the next place, we shall report the following theorem with respect to the incomplete infinite direct product of von Neumann algebras defined in [1] and [7].

THEOREM 3. *If each von Neumann algebra $\mathcal{A}_i$ acting on $\mathcal{H}_i$ has the property Q for every $i \in I$, then the $\mathfrak{C}$-adic incomplete infinite direct product $\Pi \otimes_{i \in I}^{\mathfrak{C}} \mathcal{A}_i$ has the property Q for any equivalent class $\mathfrak{C}$.*

Bures [1] proved that an incomplete infinite direct product of von Neumann algebras becomes a factor of type I, type II and type III for an appropriate equivalence class, respectively. Hence, by Theorem 3, there exists a factor of type I, type II and type III with the property $Q$ respectively. Schwartz [9] showed that there exists a factor of type III which has not the property $P$. Then, there exists a factor of type III without the property $Q$, by Theorem 1 in [2]. Therefore, by the results of [2] and the above, we have the following.

THEOREM 4. i) *The factors of type I have the property Q.* ii) *In the case of type $II_1$, type $II_\infty$ and type III, there exists a factor which has the property Q and there exists a factor which has not the property Q.*

## References


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