12. Note on PL-Homeomorphisms of Euclidean n-Space into Itself

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1. *Introduction.* Let $\mathcal{G}(n)$ be the space of all homeomorphisms of Euclidean n-space R^n into itself provided with the compact-open topology. Let $\mathcal{H}(n)$ be the subspace of all onto homeomorphisms. Let $Pl(n)$ be the subspace of all PL-homeomorphisms and $PL(n)$ be the subspace of all onto PL-homeomorphisms. Those elements in $\mathcal{G}(n)$, $\mathcal{H}(n)$, $Pl(n)$ and $PL(n)$ which preserve the origin 0 will be denoted by $\mathcal{G}_0(n)$, $\mathcal{H}_0(n)$, $Pl_0(n)$ and $PL_0(n)$ respectively. Recently Kister [1] has shown that $\mathcal{H}_0(n)$ is a weak kind of deformation retract of $\mathcal{G}_0(n)$.

In the present note we show that $PL_0(n)$ is a weak kind of deformation retract of $Pl_0(n)$. More precisely:

Theorem. *There is a continuous map F: $Pl_0(n)\times I\to Pl_0(n)$, for each n, such that*

(1) *$F(g, 0)=g$, for all g in $Pl_0(n)$,*

(2) *$F(g, 1)$ is in $PL_0(n)$ for all g in $Pl_0(n)$,*

(3) *$F(h, t)$ is in $PL_0(n)$ for all h in $PL_0(n)$, t in I.*

2. *Definitions.* Let R^n be a Euclidean n-space. We consider an ordinary triangulation on R^n. Let d be the usual metric in Euclidean n-space R^n. Let ρ be the metric in R^n defined by

$$\rho(x, y)=\max_i |x_i-y_i|,$$

for

$$x=(x_1, x_2, \cdots, x_n), \qquad y=(y_1, y_2, \cdots, y_n)$$

in R^n. The cube of side $2r$ with centre at 0 in R^n is denoted by C_r. This set is also considered as

$$C_r=\{x\in R^n \mid \rho(0, x)\leq r\}.$$

If K is a compact set in R^n containing 0, we define the *square radius* of K to be

$$r[K]=\max\{r \mid C_r\subset K\}.$$

If g_1, g_2: $K\to R^n$ are imbeddings of the compact set K, then we say g_1 and g_2 are *within* ε, if for each x in K it is true that $\rho(g_1(x), g_2(x))<\varepsilon$. If g is in $Pl_0(n)$ and K is a compact set in R^n, $V(g, K, \varepsilon)$ denotes the subset of all elements h in $Pl_0(n)$ such that $g|K$ and $h|K$ are within ε. Then the collection of all such $V(g, K, \varepsilon)$ is, of course, a base for $Pl_0(n)$.

If $0 \leq a < b < d$ and $a < c < d$ and t is in $I = [0, 1]$, then we define $\theta_t(a, b, c, d) \in PL_0(n)$ to be the PL-homeomorphism of R^n onto itself, fixed on C_a and outside C_d as follows. Let L be a ray emanating from the origin and coordinatized by distance (in the sense of metric ρ) from the origin. Then θ_t is fixed on $[0, a]$ and on $[d, \infty)$, and it takes b onto $(1-t)b + tc$ and is linear on $[a, b]$ and $[b, d]$. We denote $\theta_1(a, b, c, d)$ by $\theta(a, b, c, d)$ and $\theta(0, b, c, d)$ by $\theta(b, c, d)$. Clearly $(t; a, b, c, d) \to \theta_t(a, b, c, d)$ is continuous, regarded as a mapping from a subset of R^5 into $PL_0(n)$.

3. *A useful lemma.*

Lemma. Let g and h be in $Pl_0(n)$ with $h(R^n) \subset g(R^n)$. Let a, b, c and d be real numbers satisfying $0 \leq a < b$, $0 < c < d$ and such that $h(C_b) \subset g(C_c)$. Then there is a PL-isotopy[1] $\varphi_t(g, h; a, b, c, d) = \varphi_t$ $(t \in I)$ of R^n onto itself satisfying

1) $\varphi_0 = 1$,
2) $\varphi_1(h(C_b)) \supset g(C_c)$,
3) φ_t is fixed outside $g(C_d)$ and on $h(C_a)$,
4) $(g, h; a, b, c, d; t) \to \varphi_t$

is a continuous map from the appropriate subset of $Pl_0(n) \times Pl_0(n) \times R^5$ into $PL_0(n)$.

Proof. Let a' be $r[g^{-1} \circ h(C_a)]$; note that $a' < c$. Let b' be $r[g^{-1} \circ h(C_a)]$; note that $a' < b' \leq c < d$.

We first shrink $h(C_a)$ inside $g(C_{a'})$ with a PL-homeomorphism σ fixed outside $h(C_b)$. This can be done as follows. Let a'' be $r[h^{-1} \circ g(C_{a'})]$; note that $a'' \leq a < b$. Define

$$\sigma = \begin{cases} h \circ \theta(a, a'', b) \circ h^{-1}, & \text{on } h(C_b), \\ 1, & \text{elsewhere.} \end{cases}$$

Then σ is in $PL_0(n)$.

Next we get a PL-isotopy ψ_t $(t \in I)$ taking $g(C_{b'})$ onto $g(C_c)$, leaving $g(C_{a'})$, and the exterior of $g(C_d)$ fixed. Define

$$\psi_t = \begin{cases} g \circ \theta_t(a', b', c, d) \circ g^{-1}, & \text{on } g(C_d), \\ 1, & \text{elsewhere.} \end{cases}$$

Then ψ_t is in $PL_0(n)$.

Finally define $\varphi_t = \sigma^{-1} \circ \psi_t \circ \sigma$. Then φ_t is in $PL_0(n)$. It is easy to verify that (1), (2) and (3) are satisfied. The continuity of φ_t depends on the following three propositions.

Proposition 1. Let g be in $Pl_0(n)$, and let r and ε be two positive numbers. Then there is a $\delta > 0$ so that, if g_1 is in $V(g, C_{r+\varepsilon}, \delta)$, then

(1) $g_1(C_{r+\varepsilon}) \supset g(C_r)$,

(2) $g_1^{-1} | g(C_r)$ and $g^{-1} | g(C_r)$ are within ε.

Proposition 2. Let C be a finite complex, h: $C \to R^n$ an imbedding,

1) By *PL-isotopy* φ_t we mean an isotopy φ_t such that for each t in $[0, 1]$ φ_t is a PL-homeomorphism.

D a finite subcomplex in R^n containing $h(C)$ in its interior, and $g: D \to R^n$ another imbedding. For any $\varepsilon > 0$, there is a $\delta > 0$ so that, if $g_1: D \to R^n$, $h_1: C \to R^n$ are imbeddings within δ of g and h respectively, then $g_1 \circ h_1$ is defined and within ε of $g \circ h$.

Proposition 3. Let g and h be in $Pl_0(n)$, and let a be a non-negative number such that $h(C_a) \subset g(R^n)$. Let $r = [g^{-1} \circ h(C_a)]$. Then $r = r(g, h, a)$ is continuous simultaneously in the variables g, h and a.

These propositions are proved quite parallel with Propositions 1, 2, 3 in Kister [1].

The continuity of φ_t is easily proved by these propositions.

4. *Proof of Theorem.* Before we give the proof of Theorem we state two more propositions.

Proposition 4. Let g be in $Pl_0(n)$ and r_i be $r[g(C_i)]$ for each positive integer i. Then there is an element h in $Pl_0(n)$ such that $h(C_i) = C_{r_i}$, for each i, and h depends continuously on g.

Proposition 5. Let F: $Pl_0(n) \times [0, 1) \to Pl_0(n)$ be continuous, and denote $F(g, t)$ by g_t. Suppose $g_t | C_n = g_{1-(1/2)^n} | C_n$ for all t in $[1-(\frac{1}{2})^n, 1)$ and $n = 1, 2, \cdots$. Then F can be extended to $Pl_0(n) \times I$.

These propositions are proved quite parallel with Proposition 4, 5 in Kister [1].

We return to the proof of Theorem. Let g in $Pl_0(n)$ be given. Use Proposition 4 to find $h = h(g)$. First we shall produce a PL-isotopy α_t: $R^n \to g(R^n)$ $(t \in I)$ such that

(a) $\alpha_0 = h$,

(b) $\alpha_1(R^n) = g(R^n)$,

(c) $\alpha_t = \alpha(g, t)$ is continuous in g and t.

We do this in an infinite number of steps. To define $\alpha_t (t \in [0, \frac{1}{2}])$ we use the Lemma for $a=0$, $b=c=1$, $d=2$, and obtain φ_t $(t \in I)$. Define $\alpha_t = \varphi_{2t} \circ h$ $(t \in [0, \frac{1}{2}])$. Then α_t is in $Pl_0(n)$ for $t \in [0, \frac{1}{2}]$, $\alpha_0 = h$, $\alpha_{\frac{1}{2}}(C_1) \supset g(C_1)$ and, by Proposition 4, the Lemma, and Proposition 2, α_t $(t \in [0, \frac{1}{2}])$ is continuous in g and t. Note that $\alpha_{\frac{1}{2}}(C_2) \subset g(C_2)$ by property (3) of the Lemma.

Next we define, α_t $(t \in [\frac{1}{2}, \frac{3}{4}])$ by again using the Lemma, this time for "h" $= \alpha_{\frac{1}{2}}$, $a=1$, $b=c=2$, d$=3$, and we obtain φ_t $(t \in I)$. Now define $\alpha_t = \varphi_{4t-2} \circ \alpha_{\frac{1}{2}}$ $(t \in [\frac{1}{2}, \frac{3}{4}])$. Then α_t is in $Pl_0(n)$ for $t \in [\frac{1}{2}, \frac{3}{4}]$, α_t is an extension of that obtained in the first step, $\alpha_{\frac{3}{4}}(C_2) \supset g(C_2)$, and since $\alpha_{\frac{1}{2}}$ depends continuously on g, we can conclude as before that α_t $(t \in [\frac{1}{2}, \frac{3}{4}])$ is continuous in g and t. Note that $\alpha_{\frac{3}{4}}(C_3) \subset g(C_3)$, and that $\alpha_t | C_1 = \alpha_{\frac{1}{2}} | C_1$ for t in $[\frac{1}{2}, \frac{3}{4}]$, by property (3) of the Lemma.

We continue in this manner defining for each integer n, $\alpha_t \in Pl_0(n) (t \in [1-(\frac{1}{2})^n, 1-(\frac{1}{2})^{n+1}])$ such that $\alpha_{1-(\frac{1}{2})^n}(C_n) \supset g(C_n)$ and $\alpha_t | C_n = \alpha_{1-(\frac{1}{2})^n} | C_n$ for t in $[1-(\frac{1}{2})^n, 1-(\frac{1}{2})^{n+1}]$.

Proposition 5 allows us to define $\alpha_1 \in Pl_0(n)$ so that α_t $(t \in I)$ depends continuously on g and t, and $\alpha_1(R^n)=g(R^n)$.

In the second stage, we produce a PL-isotopy β_t: $R^n \to R^n$ $(t \in I)$ such that

(a) $\beta_0 = h$,

(b) $\beta_1 = 1$,

(c) $\beta_t = \beta(g, t)$ is continuous in g and t.

This we do again in an infinite number of steps, first obtaining β_t $(t \in [0, \frac{1}{2}])$ as follows. We have $h(C_1) = C_{r_1}$ where $r_1 = r[g(C_1)]$, since h was constructed so as to take cubes onto cubes.

We shall preserve this property throughout the PL-isotopy β_t $(t \in I)$. Let L be any ray emanating from the origin in R^n and coordinatized by distance from the origin (in the sense of metric ρ). For t in I, let φ_t take the interval $[0, r_1]$ in L linearly onto $[0, (1-t)r_1+t]$ and translate $[r_1, \infty)$ to $[(1-t)r_1+t, \infty)$. This defines φ_t in $PL_0(n)$ for each t in I. Now let $\beta_t = \varphi_{2t} \circ h$ $(t \in [0, \frac{1}{2}])$. Then β_t is in $Pl_0(n)$ for $t \in [0, \frac{1}{2}]$, $\beta_0 = h$ and $\beta_{\frac{1}{2}} | C_1 = 1$, and since r_1 and h depend continuously on g, then φ_{2t} and hence β_t are continuous in g and t.

Let s_2 be such that $\beta_{\frac{1}{2}}(C_2) = C_{s_2}$, and define β_t $(t \in [\frac{1}{2}, \frac{3}{4}])$ as follows. Let L be any ray as before, and let φ_t $(t \in I)$ take $[1, s_2]$ in L linearly onto $[1, (1-t)s_2+2t]$, translate $[s_2, \infty)$ onto $[(1-t)s_2+2t, \infty)$, and leave $[0, 1]$ fixed. Define $\beta_t = \varphi_{4t-2} \circ \beta_{\frac{1}{2}}$ $(t \in [\frac{1}{2}, \frac{3}{4}])$. Then β_t is in $Pl_0(n)$ for $t \in [\frac{1}{2}, \frac{3}{4}]$, extends β_t $(t \in [0, \frac{3}{4}])$, $\beta_{\frac{3}{4}} | C_2 = 1$, and β_t depends continuously on g and t.

Continuing this manner, as in the first stage, we obtain a PL-isotopy β_t $(t \in I)$ which depends continuously on g and t.

Now define

$$F(g, t) = \begin{cases} \alpha_{1-2t} \circ \alpha_1^{-1} \circ g, & \text{for } t \text{ in } [0, \frac{1}{2}], \\ \beta_{2t-1} \circ \alpha_1^{-1} \circ g, & \text{for } t \text{ in } [\frac{1}{2}, 1]. \end{cases}$$

Then $F(g, t)$ is in $Pl_0(n)$. It is easy to check that F satisfies (1) and (2). An immediate consequence of Proposition 4 is that h is onto if g is. Each φ_t that occurs in a step of the construction of α_t and β_t is onto, hence α_t and β_t, and finally $F(g, t)$ is onto if g is, so property (3) holds. Continuity of F follows from that of α_t and β_t and from Proposition 1 and 2.

Reference

[1] J. M. Kister: Microbundles are fibre bundles. Ann. of Math., **80**, 190-199 (1964).