50. *On Ascoli-Arzela's Theorem for Metric Space over Topological Semifield*

By Kiyoshi ISÉKI

(Comm. by Kinjirô KUNUGI, M.J.A., March 12, 1965)

In this note, we are concerned with the well known theorem of Ascoli and Arzela, and we shall generalize a recent result of K. Vala [4]. We consider a metric space X over a topological semifield K. We denote the metric by d. For the concept of topological semifields, see [1] and [2]. In our discussion, we need the concept of totally boundedness.

Definition 1. *A* subset A of a metric space X over a topological semifield K is said to be *totally bounded*, if for every neighbourhood U of 0 in K, it is possible to present it as the union of a finite number of sets with diameter less than U, in the other word, given a neighborhood U of 0 in K, there is a finite subset $\{x_k\}$ of X such that, for every $x \in f(A)$, $d(x, x_k) \in U$ for some k.

Let E be an abstract set, then a mapping $f: E \to X$ is called a *totally bounded mapping*, if $f(E)$ is totally bounded in X. All totally bounded mapping from E to X forms a metric space over K with the metric $\rho(f, g) = \sup_{x \in E} d(f(x), g(x))$. It is evident that each $\rho(f, g)$ is finite, as f, g are totally bounded mapping. This metric space will be denoted by $B_t(E, X)$. We introduce the natural topology in $B_t(E, X)$ by the topological semifield K (see [1] and [2]).

Let H be a subset of the space $B_t(E, X)$. Following K. Vala [4], we call that H has *equal variation* if for any neighborhood U of 0 in K, there is a partition $E_i (i=1, 2, \cdots, n)$ of E such that $x, y \in E_i$ $(i=1, 2, \cdots, n)$ implies $d(f(x), f(y)) \in U$ for every $f \in H$.

Then we have the following result which is a generalization of a theorem by Vala and the idea of its proof is essentially due to K. Vala [4].

Theorem 1. *Let H be a subset of $B_t(E, X)$. H is totally bounded if and only if* 1) *$H(x) = \{f(x) \mid f \in H\}$ for every $x \in E$ is totally bounded and* 2) *H has an equal variation.*

Proof. We shall suppose that H is totally bounded. Then given a neighborhood U of 0 in K, there is a finite subset $f_1, f_2, \cdots, f_n$ of H such that, for each $f \in H$, $\rho(f, f_k) \in U$ for some f_k. For any $x \in E$, we have $d(f(x), f_k(x)) \ll \rho(f, f_k) \in U$, which shows that $H(x)$ is totally bounded. Further, each f_k is totally bounded, so for each k, there is a finite partition $\{E_l^k\} (l=1, 2, \cdots, i_k)$ of E such that $x, y \in E_l^k$

implies $d(f_k(x), f_k(y)) \in U$. Consider the mixed partition $\{E_m\}$ of E by $\{E_l^k\}$. Then E_m is the form of $E_{l_1}^1 \cap E_{l_2}^2 \cap \cdots \cap E_{l_k}^k \cap \cdots \cap E_{l_n}^n$. For any f and $x, y \in E_m$, then $\rho(f, f_k) \in U$ for some k, and

$$d(f(x), f(y)) \ll d(f(x), f_k(x)) + d(f_k(x), f_k(y)) + d(f_k(y), f(y)) \in U+U+U.$$

Hence H has an equal variation.

Conversely, we shall suppose that H satisfies the conditions 1) and 2). Let U be a neighborhood of 0 in K. By the condition 2), then there is a finite partition $E_1, E_2, \cdots, E_n$ of E such that $x, y \in E_k$ implies $d(f(x), f(y)) \in U$ for every $f \in H$. Take an element $x_k \in E_k$, then, by the condition 1), $H(x_k)$ is totally bounded in X. Therefore, for each k, there is a partition $H_l^k (l=1, 2, \cdots, i_k)$ of H such that $f, g \in H_l^k$ implies $d(f(x_k), g(x_k)) \in U$. Let $\{H_m\}$ be the mixed partition of H be $\{H_l^k\}$, if $f, g \in H_m = H_{l_1}^1 \cap H_{l_2}^2 \cap \cdots \cap H_{l_k}^k$, then $f, g \in H_l^k$ and we have, for each x,

$$d(f(x), g(x)) \ll d(f(x), f(x_k)) + d(f(x_k), g(x_k)) + d(g(x_k), g(x)) \in U+U+U.$$

This shows that the diameter of each H_m is sufficently small, hence H is totally bounded. Therefore the proof is complete.

To give the other formulation, we shall consider the quasiuniformly convergence of a (directed) sequence $\{f_\alpha\}$ (see R. G. Bartle [3], p. 37). Let $\{f_\alpha\}$ be a sequence on E to X.

Definition 2. A sequence $\{f_\alpha\}$ is said to be converge to f_0 quasiuniformly on E if $f_\alpha(x) \to f_0(x)$ on E and if, for each U of 0 in K and α_0, then there is a finite set of indices $\alpha_1, \alpha_2, \cdots, \alpha_k > \alpha_0$ such that for each $x \in E$ at least one of the following inequalities is true:

$$d(f_{\alpha_i}(x), f_0(x)) \in U.$$

The concept of uniformly convergences is introduced by the usual way. We shall prove the following theorem which is a generalization of Arzela's theorem.

Theorem 2. *If a sequence $\{f_\alpha\}$ of continuous functions on a compact space E converges to a continuous function, then the convergence is quasiuniform on E. On the other hand, if the sequence $\{f_\alpha\}$ converges quasiuniformly on any topological space E, then the limit is continuous on E.*

Proof. Let f_0 be the limit of $\{f_\alpha\}$. First, if f_0 is continuous on E, then for any neighborhood U of 0 in K, index α_0 and $x_0 \in E$, there is an $\alpha = \alpha(x_0) > \alpha_0$ such that $d(f_\alpha(x_0), f_0(x_0)) \in U$. Put

$$V(x_0) = \{x \mid d(f_\alpha(x), f_0(x_0)) \in U\},$$

then $V(x_0)$ is an open set containing x_0, since f_0 and f_α are continuous. E is compact, then we can find a finite set $\{\alpha(x_i)\} (i=1, 2, \cdots, n)$ of indices which satisfies the condition of quasiuniformly convergence.

Conversely, suppose that the convergence $f_\alpha(x) \to f(x)$ is quasi-uniformly on E. For any neighborhood U of 0 in K, and $x_0 \in E$, there is an α_0 such that $\alpha > \alpha_0$ implies

$$d(f_\alpha(x_0), f_{\alpha_0}(x_0)) \in U.$$

For α_0, we can find a finite set of indices $\alpha_1, \alpha_2, \cdots, \alpha_n > \alpha_0$ satisfying the condition in Definition 2. Put $V_i = \{x \mid d(f_{\alpha_i}(x), f_{\alpha_i}(x_0)) \in U\}$. Each V_i is an open set containing x_0. The intersection V of $V_i (i=1, 2, \cdots, n)$ is open and contains x_0. If $x \in V$, then

$$d(f_0(x), f_0(x_0)) \ll d(f_0(x), f_{\alpha_i}(x)) + d(f_{\alpha_i}(x), f_{\alpha_i}(x_0)) + d(f_{\alpha_i}(x_0), f_0(x_0)) \in U + U + U.$$

for some α_i. Hence $f_0(x)$ is continuous at x_0, so $f(x)$ is continuous on E. The proof is complete.

References

[1] М. Я. Антоновский, В. Г. Болтянский, Т. А. Сарымсаков, Топологические полуполя, Ташкент (1960).

[2] М. Я. Антоновский, В. Г. Болтянский, Т. А. Сарымсаков Метрические пространства над полуполями, Ташкентского Университета 191 (1961).

[3] R. G. Bartle: On compactness in functional analysis. Trans. Amer. Math. Soc., **79**, 35-57 (1955).

[4] K. Vala: On compact sets of compact operators. Ann. Acad. Sci. Fennicae A, I 351, 1-9 (1964).