65. On Automorphisms of Abelian von Neumann Algebras

By Hisashi CHODA
Department of Mathematics, Osaka Gakugei University
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1. Throughout this note, we shall use the terminology due to J. Dixmier [2] without further explanations.

Following after H. A. Dye [3], we shall introduce some fundamental definitions on automorphisms of an abelian von Neumann algebra $\mathcal{A}$ with the faithful normal trace ϕ normalized by $\phi(1)=1$. A projection P in $\mathcal{A}$ is said to be *absolutely fixed* under an automorphism g of $\mathcal{A}$ if $Q^g=Q$ for each $Q\leqq P$. For the given two automorphisms g and h of $\mathcal{A}$, we shall denote by $F(g, h)$ the maximal projection in $\mathcal{A}$ which is absolutely fixed under gh^{-1}.

Let G be a group of ϕ-preserving automorphisms of $\mathcal{A}$;

$$\phi(A^g)=\phi(A) \quad \text{for each } A\in\mathcal{A} \text{ and } g\in G.$$

If $F(g, 1)=0$ for each $g\neq 1$ in G, then G is called *freely acting*. If α is an automorphism of $\mathcal{A}$, we say that α *depends* on G if $\text{l.u.b}_{g\in G} F(\alpha, g)=1$. We shall denote by $[G]$ the collection of all automorphisms of $\mathcal{A}$ which preserve ϕ and depend on G. We shall call $[G]$ the *full group* determined by G.

In this paper, we shall give a characterization of dependence of an automorphism with respect to the given group G in terms of the crossed product of an abelian von Neumann algebra $\mathcal{A}$.

2. At first we shall review briefly the concept of the crossed product of an abelian von Neumann algebra by an enumerable freely acting group G of ϕ-preservin automorphisms of $\mathcal{A}$, cf. [1], [4], and [5].

We shall denote an operator valued function defined on G by $\sum_{g\in G} g\otimes A_g$ where $A_g\in\mathcal{A}$ is the value of the function at $g\in G$. Let $\mathcal{D}$ be the set of all functions such that $A_g=0$ up to a finite subset of G. Then $\mathcal{D}$ is a linear space with the usual operations of the addition and the scalar multiplication, and becomes a $*$-algebra by the following operations:

$$\Big(\sum_{g\in G} g\otimes A_g\Big)\Big(\sum_{h\in G} h\otimes B_h\Big)=\sum_{g,h\in G} gh\otimes A_g B_h^{g^{-1}}$$

and

$$\Big(\sum_{g\in G} g\otimes A_g\Big)^*=\sum_{g\in G} g^{-1}\otimes A_g^{*g}.$$

For a trace ϕ in $\mathcal{A}$, we shall introduce a trace φ in $\mathcal{D}$ by

$$\varphi(g\otimes A_g)=\begin{cases}\phi(A_g) & \text{for } g=1,\\ 0 & \text{for } g\neq 1,\end{cases}$$

and

$$\varphi(\sum_{g\in G} g\otimes A_g)=\sum_{g\in G}\varphi(g\otimes A_g).$$

Then the restriction of φ on $\mathcal{A}=1\otimes\mathcal{A}$ coincides with ϕ and φ is faithful on $\mathcal{D}$, cf. [4]. Let $\mathcal{H}$ be the representation space of $\mathcal{A}$ by ϕ, cf. [2], then $G\otimes\mathcal{H}$, in the sense of H. Umegaki [6], is the representation space of $\mathcal{D}$ by φ, and $\mathcal{D}$ is represented faithfully on $G\otimes\mathcal{H}$.

We define the operators $1\otimes A$ and U_g on $G\otimes\mathcal{H}$ for each $A\in\mathcal{A}$ and $g\in G$ by

$$1\otimes A(\sum_{h\in G} h\otimes B_h)=\sum_{h\in G} h\otimes AB_h$$

and

$$U_g(\sum_{h\in G} h\otimes B_h)=\sum_{h\in G} gh\otimes B_h^{g^{-1}},$$

for any $\sum_{h\in G} h\otimes B_h\in\mathcal{D}$, being considered as a dense linear subset of $G\otimes\mathcal{H}$. Then U_g is a unitary operator and we have

$$U_g^*(1\otimes A)U_g=1\otimes A^g.$$

Hereafter, we shall identify $1\otimes A$ with A since $\mathcal{A}$ is isomorphic to $1\otimes\mathcal{A}$.

The *crossed product* $G\otimes\mathcal{A}$ of $\mathcal{A}$ by G (with respect to ϕ) is the weak closure of $\mathcal{D}$ on $G\otimes\mathcal{H}$, being considered $\mathcal{D}$ as a *-algebra of operators on $G\otimes\mathcal{H}$, that is, $G\otimes\mathcal{A}$ is the von Neumann algebra generated by $\mathcal{A}$ and $\{U_g : g\in G\}$. Then each element in $G\otimes\mathcal{A}$ has the form of $\sum_{g\in G} A_g U_g$, where $A_g\in\mathcal{A}$.

Now, we shall investigate the interrelation of the dependence of automorphisms and the crossed product of abelian von Neumann algebras in the following

THEOREM 1. *Let $\mathcal{A}$ be an abelian von Neumann algebra with the faithful normal trace ϕ normalized by $\phi(1)=1$, G be a freely acting group of ϕ-preserving automorphisms of $\mathcal{A}$ and α be an automorphism of $\mathcal{A}$ which depends on G. Then α can be extended to an inner automorphism of $G\otimes\mathcal{A}$ which is induced by a unitary operator*

$$U=\sum_{g\in G} E_g U_g,$$

where E_g satisfies the following properties:

(1) *E_g is a projection in $\mathcal{A}$ for every $g\in G$,*

(2) *$E_g E_h=0$ for $g\neq h$,*

(3) *$\sum_{g\in G} E_g=1$,*

(4) *E_g is absolutely fixed under αg^{-1}.*

Proof. Put

$$E_g=F(\alpha, g) \quad\text{and}\quad U=\sum_{g\in G} E_g U_g,$$

then it is clear by the definition of $F(\alpha, g)$ that E_g satisfies the conditions (1) and (4).

Since G is a freely acting group,

$$E_g E_h = F(\alpha, g) F(\alpha, h) = 0,$$

that is (2). By the dependence of α,

$$\sum_{g \in G} E_g = \sum_{g \in G} F(\alpha, g) = 1,$$

which is (3).

By the following direct computations, we can see that U is a unitary operator in $G \otimes \mathcal{A}$ and that U induces an inner automorphism of $G \otimes \mathcal{A}$ which is an extension of α:

$$\begin{aligned} U^*U &= (\sum_{g \in G} E_g U_g)^* (\sum_{h \in G} E_h U_h) = \sum_{g,h \in G} U_g^* E_g E_h U_h \\ &= \sum_{g \in G} U_g^* E_g U_g = \sum_{g \in G} E_g^{\alpha} = 1, \\ UU^* &= (\sum_{g \in G} E_g U_g)(\sum_{h \in G} E_h U_h)^* = \sum_{g,h \in G} E_g U_g U_h^* E_h \\ &= \sum_{g,h \in G} E_g E_h^{hg^{-1}} U_{gh^{-1}} = \sum_{g,h \in G} (E_g E_h)^{\alpha g^{-1}} U_{gh^{-1}} \\ &= \sum_{g \in G} E_g = 1, \end{aligned}$$

and

$$\begin{aligned} U^*AU &= (\sum_{g \in G} E_g U_g)^* A (\sum_{h \in G} E_h U_h) = \sum_{g,h \in G} U_g^* E_g A E_h U_h \\ &= \sum_{g \in G} U_g^* E_g A U_g = \sum_{g \in G} (E_g A)^{\alpha} = A^{\alpha}, \end{aligned}$$

for each $A \in \mathcal{A}$. This proves the theorem.

Conversely, we have the following

THEOREM 2. *Let $\mathcal{A}$ and G be as in Theorem 1. Then a ϕ-preserving automorphism α of $\mathcal{A}$ depends on G if α can be extended to an inner automorphism of $G \otimes \mathcal{A}$.*

Proof. We suppose that α can be extended to an inner automorphism of $G \otimes \mathcal{A}$ which is induced by a unitary operator U in $G \otimes \mathcal{A}$. Then we have

$$A^{\alpha} = U^*AU, \quad \text{for each } A \in \mathcal{A},$$

whence $UA^{\alpha} = AU$. Set $U = \sum_{g \in G} A_g U_g$, then, for any $A \in \mathcal{A}$,

$$UA^{\alpha} = (\sum_{g \in G} A_g U_g) A^{\alpha} = \sum_{g \in G} A_g A^{\alpha g^{-1}} U_g$$

and

$$AU = A(\sum_{g \in G} A_g U_g) = \sum_{g \in G} A A_g U_g.$$

therefore we have

$$A_g A^{\alpha g^{-1}} = A A_g,$$

for each $A \in \mathcal{A}$ and $g \in G$.

Let E_g be the carrier projection of A_g. Then, for any character χ in E_g (that is a homomorphism of $\mathcal{A}$ onto the field of all complex numbers such that $\chi(E_g) = 1$),

$$\chi(A_g)\chi(A^{\alpha g^{-1}}) = \chi(A_g A^{\alpha g^{-1}}) = \chi(AA_g) = \chi(A)\chi(A_g),$$

for each $A \in \mathcal{A}$, so that we have

$$\chi(A^{\alpha g^{-1}}) = \chi(A), \quad \text{for each } A \in \mathcal{A}.$$

Therefore E_g is absolutely fixed under αg^{-1}, so that E_g is dominated by $F(\alpha, g)$.

Denote $E = \text{l.u.b.}_{g \in G} E_g = \sum_{g \in G} E_g$ and $F = 1 - E$. Then

$$FU = F(\sum_{g \in G} A_g U_g) = \sum_{g \in G} F A_g U_g = 0,$$

whence $E = 1$, or $\text{l.u.b.}_{g \in G} F(\alpha, g) = 1$. Therefore α depends on G.

3. We shall call the algebra of all operators of $\mathcal{A}$ which is invariant under all $g \in G$ the *fixed algebra* of G.

THEOREM 3. *Let $\mathcal{A}$ and G be as Theorem 1. Then $[G]$ has the same fixed algebra as G.*

Proof. Let $\mathcal{Z}$ be the fixed algebra of G. Then, for each $A \in \mathcal{Z}$,

$$\begin{aligned} A^{\alpha} &= U^*AU = (\textstyle\sum_{g\in G} E_g U_g)^* A (\sum_{h\in G} E_h U_h) \\ &= \textstyle\sum_{g,h\in G} U_g^* E_g A E_h U_h = \sum_{g\in G} U_g^* E_g A U_g \\ &= \textstyle\sum_{g\in G} (E_g A)^g = \sum_{g\in G} E_g^{\alpha} A = A. \end{aligned}$$

Therefore the fixed algebra of $[G]$ contains that of G. The converse implication is obvious. This proves the theorem.

References

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