63. Contraction of the Group of Diffeomorphisms of R^n

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In this note, we show that the group of all diffeomorphisms of class $C^r(1\leq r\leq\infty)$ of R^n is contractible to $O(n)$ under the $C^{r'}$-topology. $(1\leq r'\leq r)$.

The group of diffeomorphisms. Let $f:R^n\to R^n$ be a diffeomorphism of class C^r and set

$$f(x)=(f_1(x),\cdots,f_n(x)) \qquad (x\in R^n),$$

where each $f_i(x)$ is a C^r-function on R^n. Furthermore, we set

$$|f(x)|=\sqrt{\sum_i|f_i(x)|^2},$$

$$D^pf(x)=(D^pf_1(x),\cdots,D^pf_n(x)),\ D^p=\frac{\partial^{|p|}}{\partial x^{i_1}\cdots\partial x^{i_n}},$$

$$p=(i_1,\cdots,i_n),\ |p|=i_1+\cdots+i_n,$$

$$J(f)(x)=\frac{\partial(f_1,\cdots,f_n)}{\partial(x_1,\cdots,x_n)}.$$

The set of all C^r-diffeomorphisms of R^n forms a group. For any $\varepsilon>0$ and an compact set K of R^n, consider the following subset of this group:

$U(f,K,\varepsilon)=\{g\,||f(x)-g(x)|<\varepsilon,\ |D^pf(x)-D^pg(x)|<\varepsilon,\ |p|\leq r',\ x\in K\}$,
where $i\leq r'\leq r$.

Taking these $U(f,K,\varepsilon)$ as the open basis, the group of all C^r-diffeomorphisms becomes a topological group. (Cerf [1], 1, 4, 2. Proposition 2, 4°. (p. 287)). We denote this group by $H^{r,r'}(n)$ and denote the subgroup of $H^{r,r'}(n)$ formed by those diffeomorphisms fixing the origin by $H_0^{r,r'}(n)$. The contraction $\rho:H^{r,r'}(n)\times I\to H^{r,r'}(n)$ defined by

$$\rho(f,t)=(f_1(x)-tf_1(0),\cdots,f_n(x)-tf_n(0)),$$

shows that $H_0^{r,r'}(n)$ is a strong deformation retract of $H^{r,r'}(n)$. Hence in the remainder, we consider the group $H_0^{r,r'}(n)$.

Homomorphisms J_0 and ι. Set

$$J_0(f)=J(f)(0),\ f\in H_0^{r,r'}(n),$$

$$\iota(a_{ij})=\left(\sum_i a_{i1}x_i,\cdots,\sum_i a_{in}x_i\right),\ (a_{ij})\in GL(n,R).$$

Then, for

$$U((a_{ij}),\varepsilon)=\left\{(b_{ij})\,|\,\sqrt{\sum_{ij}(a_{ij}-b_{ij})^2}<\varepsilon\right\},$$

we have

$$J_0(U(f,K,\varepsilon/n))\subset U(J_0(f),\varepsilon),\quad \text{if } 0\in K,$$

$$(U(a_{ij}), \varepsilon/M)) \subset U(\iota(a_{ij}), K, \varepsilon), \quad \text{if } M = \max_{x \in K} |x|.$$

Therefore the maps $J_0 : H_0^{r,r'}(n) \to GL(n, R)$ and $\iota : GL(n, R) \to H_0^{r,r'}(n)$ are both continuous. We note that J_0 is not continuous if we use the compact open topology.

Cleary, ι is an into isomorphism and its image is a closed subgroup of $H_0^{r,r'}(n)$. Hence we identify $\iota(GL(n, R))$ with $GL(n, R)$. $J_0\iota$ is the identity map of $GL(n, R)$.

Lemma 1. Let h be a C^r-function on R^n with $h(0)=0$, and set

$$h(x, t) = t^{-1}h(tx_1, \cdots, tx_n), \quad 0 < t \leq 1,$$
$$h(x, 0) = \sum_i \frac{\partial h}{\partial x_i}(0)x_i,$$

then $h(x, t)$ and $D^p h(x, t)(|p| \leq r)$ are continuous as the functions on $R^n \times I$.

Proof. If $t \neq 0$, the continuity follows from the definition. As $h(0)=0$, we have by the theorem of mean value,

$$h(x, t) = \sum_{i=1} \frac{\partial h}{\partial x_i}(\theta tx)x_i, \quad 0 < \theta < 1.$$

As each $\partial h/\partial x_i$ is continuous, setting

$$\rho_{i,t}(x) = \max_{0 \leq s \leq t} \left| \frac{\partial h}{\partial x_i}(sx) - \frac{\partial h}{\partial x_i}(0) \right|,$$

$\rho_{i,t}(x)$ is continuous in t and tends to 0 if t tends to 0. This proves the continuity of h at $t=0$, because we get

$$|h(x, t) - h(x, 0)| \leq \sum \rho_{i,t}(x)|x_i|.$$

The continuity of $D^p h(x, t)$ follows from the following equality:

(1) $$D^p(h(x, t)) = t^{|p|-1}(D^p h)(tx_1, \cdots, tx_n).$$

Definition 1. For $f \in H_0^{r,r'}(n)$, define $f_t : R^n \to R^n$ by

$$f_t(x) = t^{-1}f(tx)$$
$$= (t^{-1}f_1(tx_1, \cdots, tx_n), \cdots, t^{-1}f_n(tx_1, \cdots, tx_n)), \quad 0 < t \leqq 1.$$
$$f_0(x) = J_0(f)(x) = \left(\sum_i \frac{\partial f_1}{\partial x_i}(0)x_i, \cdots, \sum_i \frac{\partial f_n}{\partial x_i}(0)x_i\right).$$

Lemma 2. f_t has following properties:

(i) $f_1 = f$ and $f_0 \in (GL(n, R))$.

(ii) The correspondence $f \to f_t$ is a homomorphism for all t.

(iii) Each f_t belongs to $H_0^{r,r'}(n)$.

(iv) As the maps of $R^n \times I$ to R^n, the maps $g, h_p : R^n \times I \to R^n$ defined by $g(x, t) = g_t(x)$ and $h_p(x, t) = (D^p f_t)(x)$ are all continuous for $|p| \leq r$.

(v) If f belongs to $\iota(GL(n, R))$ then $f_t = f$ for all t.

Proof. (i) follows from the definition, and (ii) follows from

(2) $$f_t g_t(x) = t^{-1}f(t(t^{-1}g(tx)))$$
$$= t^{-1}f(g(tx)) = (fg)_t(x).$$

By (2), $f_t f_t^{-1}(x)=x$, hence f_t is a homeomorphism of R^n, and as we get

$$J(f_t)(x)=J(f)(tx),$$

for all t (containing $t=0$), we obtain (iii).

(iv) follows from lemma 1. (v) is clear by the definition.

Define the map $\Phi: H_0^{r,r'}(n)\times I \to H_0^{r,r'}(n)$ by

$$\Phi(f, t)=f_t.$$

By (i) and (v) of lemma 2, we obtain

$$(3) \qquad \Phi(f, 1)=f,\ \Phi(f, 0)\in \iota(GL(n, R)),$$

$$(4) \qquad \Phi(f, t)=f, \quad \text{for all } t, \text{ if } f\in \iota(GL(n, R)).$$

Continuity of Φ. Let K be an arbitrary compact set in R^n and set $M=\max_{x\in K}|x|$. Furthermore, set

$$\hat{K}=\bigcup_{0\le t\le 1} tK.$$

Then since $\hat{K}$ is a continuous image of the compact set $K\times I$, $\hat{K}$ is compact.

Lemma 3. If g belongs to $U(f, K, \varepsilon)$, then we have

$$(5) \qquad |D^p f_t(x)-D^p g_t(x)|<\varepsilon, \quad \text{if } x\in K, p\le r',$$

$$(6) \qquad |f_t(x)-g_t(x)|<n\sqrt{n}M, \quad \text{if } x\in K,$$

for all t $(0\le t\le 1)$.

Proof. By (1), we get

$$\begin{aligned} &|D^p f_{i,t}(x)-D^p g_{i,t}(x)| \\ &\quad = t^{|p|-1}|(D^p f_i)(tx)-(D^p g_i)(tx)| \\ &\quad \le |(D^p f_i)(y)-(D^p g_i)(y)|, \end{aligned}$$

where $x\in K$ and $y=tx\in \hat{K}$. Hence we obtain (5).

By the mean value theorem, we have

$$\begin{aligned} |f_{i,t}(x)-g_{i,t}(x)| &= \left|\sum_j \frac{\partial}{\partial x_j}(f_i-g_i)(\theta tx)x_j\right| \\ &\le \sum_j \left|\frac{\partial f_i}{\partial x_j}(\theta tx)-\frac{\partial g_i}{\partial x_j}(\theta tx)\right| |x_j| \\ &\le nM\varepsilon. \qquad\qquad 0<\theta<1, x\in K. \end{aligned}$$

Therefore we get (6).

Lemma 4. Φ is continuous.

Proof. As f_t and $D^p f_t$ are continuous on $R^n\times I$, we can choose for any compact set K, $t_0\in I$ and $\varepsilon'>0$, a positive number α satisfying

$$(7) \qquad f_t\in U(f_{t_0}, \hat{K}, \varepsilon'), \quad \text{if } |t-t_0|<2\alpha.$$

In (7), we take ε' to be smaller than $\min.(\varepsilon/2n\sqrt{n}M, \varepsilon/2)$ for given ε. Then if g belongs to $U(f, \hat{K}, \varepsilon')$, it follows from lemma 3 and (5) that

$$\begin{aligned} &|f_{t_0}(x)-g_t(x)| \\ &\quad \le |f_{t_0}(x)-f_t(x)|+|f_t(x)-g_t(x)| \\ &\quad \le \varepsilon'+n\sqrt{n}M\varepsilon'<\varepsilon, \qquad x\in K, |t-t_0|<\alpha, \end{aligned}$$

and

$$
\begin{aligned}
&|D^p f_{t_0}(x) - D^p g_t(x)| \\
&\quad \leq |D^p f_{t_0}(x) - D^p f_t(x)| + |D^p f_t(x) - D^p g_t(x)| \\
&\quad \leq \varepsilon' + \varepsilon' < \varepsilon, \qquad\qquad x \in K, \ |t - t_0| < \alpha.
\end{aligned}
$$

Therefore we get

$$\Phi(U(f, \hat{K}, \varepsilon') \times (I \cap (t_0 + \alpha, t_0 - \alpha))) \subset U(f_{t_0}, K, \varepsilon), \tag{8}$$

for arbitrary $t_0 \in I$, K and ε. Hence Φ is continuous.

As Φ is continuous, we get by (3) and (4) the following

Theorem. $\iota(GL(n, R))$ is a strong deformation retract of $H_0^{r,r'}(n)$.

As $\iota(GL(n, R))$ is isomorphic to $GL(n, R)$ (as a topological group) and $GL(n, R)$ is contractible to $O(n)$, this theorem proves our assertion.

Reference

[1] Cerf, J.: Topologie de certains espaces de plongements. Bull. Soc. math. France, **89**, 227-380 (1961).