# 83. A New Interpretation of Gottschalk's Results on Almost Periodicity

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(Comm. by Kinjirô KUNUGI, M.J.A., May 19, 1965)

Let $X$ be a metric space over a topological semifield $R$. The concept of such a metric space was introduced in M. Ya. Antonovski, V. G. Boltjanski and T. A. Sarimsakov ((1) and (2)). Let $T$ be a topological group, and let $f(x, t)$ be a mapping from $X \times T$ to $X$. We often use two conventional notations $f^t(x)$, $f^x(t)$ instead of $f(x, t)$. For the family $\{f(x, t)\}$, we introduce two concepts.

Definition 1. $\{f^t(x)\}$ is said to be *equicontinuous* at $x$ of $X$, if for every neighborhood $U$ of $0$ in $R$, there is a neighborhood $V$ of $0$ in $R$ such that $y \in \Omega(x, V)$ implies $\rho(f^t(x), f^t(y)) \in U$ for all $t \in T$.

If $\{f^t(x)\}$ is equicontinuous at each point of $X$, $\{f^t(x)\}$ is said to be *equicontinous* on $X$.

Definition 2. $\{f^t(x)\}$ is said to be *equiuniformly continuous*, if for every neighborhood $U$ of $0$ in $R$, there is a neighborhood $V$ of $0$ in $R$ such that $y \in \Omega(x, V)$ implies $\rho(f^t(x), f^t(y)) \in U$.

Of course $\rho$ denotes the metric on $X$.

The set $f(x, T)$ is called the *orbit* of $x$. Following W.H. Gottschalk (3), we shall define that $x$ of $X$ is almost periodic.

A subset $D$ of $T$ is said to be *relatively dense*, if there is a compact set $A$ of $T$ such that each left translate of $A$ meets $D$, i.e. $T=DA$. A point $x$ of $X$ is said to be *almost periodic*, if for every neighborhood $\Omega(x, U)$ of $x$ there is a relatively dense set $D$ in $T$ such that $f(x, D) \subset \Omega(x, U)$.

Then we have the following two propositions on almost periodicity which is obtained by W.H. Gottschalk (3). We shall suppose that $f^t(x)$ is a transformation group on $X$.

Theorem 1. *If the family $\{f^t(x)\}$ is equiuniformly continuous, and the orbit of $x$ is totally bounded, then $x$ is almost periodic.*

Proof. For a neighborhood $U$ of $0$ in $R$, there is a neighborhood $V$ of $0$ in $R$ such that $\rho(x, y) \in V$ implies $\rho(f^t(x), f^t(y)) \in U$.

$f(x, T)$ is totally bounded, then we can find a finite set $t_1, t_2, \cdots, t_n$ of $T$ such that $f(x, T) \bigcup_{i=1}^{n} \Omega(f^{t_i}(x), V)$. Hence for $t \in T$, there is at least one $t_i$ for which $f^t(x)=f(x, t) \in \Omega(f^{t_i}(x), V)$. So we have $\rho(f^{tt_i^{-1}}(x), x) \in U$, and then $f^{tt_i^{-1}}(x) \in \Omega(x, U)$. Therefore $x$ is almost periodic.

Theorem 2. *If the family $\{f^t(x)\}$ is equicontinuous at $x$, $f^x(t)$*

*is continuous on $T$ and $x$ is almost periodic, then the orbit of $x$ is totally bounded.*

Proof. To prove that the orbit of $x$ is almost periodic, we take any neighborhood $\Omega(x, U)$ of $x$. For $U$, there is a neighborhood $V$ of 0 in $R$ such that $U \supset V+V$. Furthermore, for $V$, we can find a neighborhood $W$ of 0 in $R$ such that $y \in \Omega(x, W)$ implies $\rho(f^t(x), f^t(y)) \in W$ for all $t \in T$. By the hypothesis, there are a set $D$ of $T$ and a compact set $A$ of $T$ such that $T=DA$ and $f(x, D) \subset \Omega(x, W)$. Consider the family $\{f^t(x)\}$, then we have

$$f(x, T)=f(f(x, D), A) \subset f(\Omega(x, W), A)$$
$$\subset \Omega(f(x, A), V)=\bigcup_{t\in A}\Omega(f^x(t), V) .$$

On the other hand, $f^x(t)$ is continuous on $T$, so $C=\bigcup_{t\in A} f^x(t)$ is a compact set. Hence $f(x, T)$ is contained in the $V$-neighborhood of the compact set $C=\bigcup_{t\in A} f^x(t)$. $\{\Omega(f^x(t), V)\}$ is an open covering of $C$, then there is a finite set $t_1, t_2, \cdots, t_n$ such that $\Omega(f^x(t_i), V)$ $(i=1, 2, \cdots, n)$ covers $C$. Consider the family of open sets $\Omega(f^x(t_i), U)$ $(i=1, 2, \cdots, n)$. Take an element $t \in T$, then there is $t_i$ such that $\rho(f^x(t), f^x(t_i)) \in V$. For $y \in \Omega(f^x(t), V)$, we have $\rho(y, f^x(t)) \in V$, hence $\rho(y, f^x(t_i)) \in V+V \subset U$. This shows $y \in \Omega(f^x(t_i), U)$. Therefore we have

$$f(x, T) \subset \bigcup_{t\in A}\Omega(f^x(t), V) \subset \bigcup_{i=1}^{n}\Omega(f^x(t_i), U) ,$$

so $f(x, T)$ is totally bounded. The proof is complete.

These two theorems are an interpretation of W.H. Gottschalk's results from the view of topological semifield.

Next we shall consider Gottschalk theorem (3) which is a generalization of Arzela theorem from the standpoint of topological semifields.

Let $X$ and $Y$ be two metric spaces over topological semifields $R_1$ and $R_2$ respectively and let $F$ be a family of functions $f(x)$ from $X$ to $Y$. We introduce a topology on $F$. Let $f \in F$, we define a neighborhood $U(f)$ of $f$ by $U(f)=\{g \mid \rho(f(x), g(x)) \in U$ for all $x$ of $X\}$, where $U$ is any neighborhood of 0 in $R_2$. If $Y$ is totally bounded, the topology gives a metric on $F$. The metric $\rho$ is defined by

$$\rho(f, g)=\sup_{x\in X}\rho(f(x), g(x))$$

for $f, g \in F$. Then $F$ is a metric space over the topological semifield $R_2$. Then we have the following

Theorem 3. *If $X$ and $Y$ are totally bounded and $F$ is equiuniformly continuous, then $F$ is totally bounded.*

Proof. Let $U$ be a neighborhood of 0 in $R_2$. Take neighborhoods $V, W$ such that $V+V \subset U$ and $W+W \subset V$. We make a $W$-net of $Y$ and let $y_1, y_2, \cdots, y_n$ be the centers of these nets. Therefore, if

$y \in Y$, we can find $\Omega(y_i, W)$ for which $y \in \Omega(y_i, W)$. Then $\Omega(y, W) \subset \Omega(y_i, V)$. The family $F$ is equiuniformly continuous, so for $W$, there is a neighborhood $W_1$ of 0 in $R_1$ such that $\rho(x, y) \in W_1$ implies $\rho(f(x), f(y)) \in W$ for all $x, y \in X$ and $f \in F$. Let $x_1, x_2, \cdots, x_m$ be the centers of some $W_1$-net of $X$, i.e. $X = \bigcup_{i=1}^{m} \Omega(x_i, W_1)$. Take a function $f(x)$ of $F$, then for each $x_i$, there is at least one $y_i$ such that $\Omega(x_i, W_1) \ni y$ implies $f(y) \in \Omega(y_j, V)$. Each $f \in F$ and each $x_i$ induce a correspondence from $x_i (i=1, 2, \cdots, m)$ to some $y_j (j=1, 2, \cdots, m)$. Following the correspondence, $F$ is divided into a finite number of subsets $F_1, F_2, \cdots, F_p$, where any two of these subsets are not necessary disjoint. Take a function $f_i(x)$ from each $F_i$, then we have $F = \bigcup_{i=1}^{p} U(f_i)$. Hence $F$ is totally bounded.

Theorem 4. *If the family $F$ is totally bounded and every function of $F$ is uniformly continuous, then $F$ is equiuniformly continuous.*

Proof. To prove that $F$ is equiuniformly continuous, take a neighborhood $U$ of 0 in $R_2$. Let $V$ be a neighborhood of 0 in $R_2$ such that $V+V+V \subset U$. $F$ is totally bounded, then we can take the centers $f_1, f_2, \cdots, f_n$ of the $V$-net of $F$, i.e. $F = \bigcup_{i=1}^{n} V(f_i)$. For each $f_i$, we select a neighborhood $V_i$ of 0 in $R_1$ such that $\rho(x, y) \in V_1$ implies $\rho(f_i(x), f_i(y)) \in V$. Put $W = \bigcap_{i=1}^{n} V_i$, then $W$ is also a neighborhood of 0 in $R_1$. For $x, y$ for which $\rho(x, y) \in W$ and any $f$ of $F$, if $f \in U(f_i)$, then we have

$$\rho(f(x), f(y)) \ll \rho(f(x), f_i(x)) + \rho(f_i(x), f_i(y)) + \rho(f_i(y), f(y)) \in V+V+V \subset U .$$

Therefore, $F$ is totally bounded.

Now we shall discuss a result on almost periodic transformation group by G.H. Gottschalk (3), but the essence of Theorem is not differ, as we consider it in a *compact* metric space over a topological semifield.

Let $f^t(x)$ be a transformation group on $X$. If for a neighborhood $U$ of 0 in $R$ there is a compact set $A$ of $T$ such that for each $t$ of $T$ we can find $s$ of $A$ for which $\rho(f^t(x), f^s(x)) \in U$ for all $x$, $f^t(x)$ is called *almost periodic transformation.*

Theorem 5. *Let $X$ be a metric space over a topological semifield $R$. If $X$ is compact and if $f^t(x)$ is continuous on $X \times T$, then the following propositions are equivalent:*

1) *$\{f^t(x)\}$ is totally bounded by the topology mentioned above.*
2) *$\{f^t(x)\}$ is equiuniformly continuous.*
3) *$f^t(x)$ is almost periodic transformation.*

Proof. It is evident that the first two propositions are equivalent by the results above.

Assume the proposition 2), then for a neighborhood $U$ of 0 in $R$, there is a neighborhood $V$ of 0 in $R$ such that $\rho(x, y) \in V$ implies $\rho(f^t(x), f^t(y)) \in U$ for all $x, y$ and $t \in T$. $\{f^t(x)\}$ is totally bounded, hence there is a $V$-net of $\{f^t(x)\}$, i.e. $\{f^t(x)\} = \bigcup_{i=1}^{n} V(f^{t_i})$ for some $t_1, t_2, \cdots, t_n \in T$. Take $t \in T$, then for some $i$, we have $\rho(f^t(x), f^{t_i}(x)) \in V$ for all $x \in X$. Hence $\rho(f^{tt_i^{-1}}(x), x) \in U$. This implies that $\{f^t\}$ is almost periodic, and we have 2)$\rightarrow$3).

Assume the proposition 3). For a neighborhood $U$ of 0 in $R$, we take a neighborhood $V$ for which $V+V+V \subset U$. For $V$, there is a compact set $A$ of $T$ such that for $t \in T$ we can find $s \in A$ for which $\rho(f^t(x), f^s(x)) \in V$ for all $x \in X$. Clearly $f^t(x)$ is uniformly continuous on $X \times A$, hence for $V$ there is a neighborhood $W$ of 0 in $R$ such that $\rho(x, y) \in W$ implies $\rho(f^s(x), f^s(y))$ for all $x, y \in W$ and $s \in A$. Therefore, if $\rho(x, y) \in W$, then for some $s$,

$$\rho(f^t(x), f^t(y)) \ll \rho(f^t(x), f^s(x)) + \rho(f^s(x), f^s(y)) + \rho(f^s(y), f^t(y)) \subset V + V + V \subset U\,.$$

This shows that $\{f^t\}$ is equiuniformly continuous. The proof is complete.

## References


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