116. Tauberian Theorems for Cesàro Sums. I

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1. **Introduction.** Let s_n and S_n^α be partial sum and Cesàro sum of order $\alpha\,(\alpha>-1)$, of a series $\sum_{n=0}^{\infty} a_n$ respectively. It is well-known that $S_n^\alpha=\sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} s_\nu$, where $A_\nu^{\alpha-1}$ is a coefficient of x^ν in $(1-x)^{-\alpha}$ $(|x|<1)$, and let σ_n^α be the Cesàro mean of the series $\sum_{n=0}^{\infty} a_n$, that is, $\sigma_n^\alpha=S_n^\alpha/A_n^\alpha$.

The series $\sum_{n=0}^{\infty} a_n$ is said to be (C, α)-summable $(\alpha>-1)$ to s if $\sigma_n^\alpha\to s$ as $n\to\infty$.

A. L. Dixon and W. L. Ferrar [2] proved the following theorem.

THEOREM A. If $W(x)$ and $V(x)$ are positive increasing functions of $x>0$, and $S_n^\delta=o(W(n))$, $(\delta>0)$, $s_n=S_n^0=O(V(n))$, then

$$S_n^\gamma=o[(V(n))^{1-\gamma/\delta}\,(W(n))^{\gamma/\delta}],\ (0<\gamma<\delta).$$

This theorem was generalized by K. Kanno [4] making use of the L. S. Bosanquet's method [1], as follows.

THEOREM B. Let $W(x)$ and $V(x)$ be positive functions of $x>0$ and satisfy the following conditions:

(1.1) $\begin{cases} \text{(i) there exists a real number } \beta>0 \text{ such that } n^\beta V(n) \text{ is non-decreasing;} \\ \text{(ii) } W(n) \text{ is non-decreasing;} \\ \text{(iii) } W(n)=O(V(n)) \text{ as } n\to\infty. \end{cases}$

And if

$$s_n=O(n^\beta V(n)) \tag{1.2}$$

and

$$S_n^\delta=o(n^\alpha W(n)) \text{ as } n\to\infty, \tag{1.3}$$

where $\delta+\beta\geqq\alpha>-1$, then

$$S_n^\gamma=o[n^{(\delta-\gamma)\beta/\delta+\alpha\gamma/\delta}\,(V(n))^{1-\gamma/\delta}\,(W(n))^{\gamma/\delta}],\ (0<\gamma<\delta). \tag{1.4}$$

M. S. Rangachari [5] tried to generalize Theorem A. However, it seems to me that his proposition is lacking in one condition. Our attempt here is to add the condition (1.5) (iv) to his. This is the following Theorem 1.

THEOREM 1. Let $W(x)$ and $V(x)$ be positive functions of $x>0$, such that

$$
(1.5)\quad \begin{cases} \text{(i)} & W(x) \text{ is non-decreasing and unbounded, there exist two} \\ & \text{positive constants } H \text{ and } \eta (0<\eta<1) \text{ for which} \\ \text{(ii)} & V(x')/V(x)<H \text{ if } 0<|x'-x|<\eta x, \\ \text{iii)} & (W(x)/V(x))^{1/\delta}=O(x) \text{ as } x\to\infty \text{ where } 0<\delta\leqq 1 \\ & \text{and} \\ \text{(iv)} & W(x)/V(x)\to\infty \text{ as } x\to\infty. \end{cases}
$$

Then

$$(1.6)\qquad S_n^\delta=o(W(n)) \text{ as } n\to\infty$$

and

$$(1.7)\qquad s_n=O(V(n)) \text{ as } n\to\infty,$$

together imply, for any γ such that $0<\gamma<\delta$,

$$(1.8)\qquad S_n^\gamma=o[(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}] \text{ as } n\to\infty.$$

The proof of this theorem may be obtained by the same method of the M. S. Rangachari's paper [5].

Next Theorem 2 is the generalization of Theorem A, Theorem B, and Theorem 1, in the proof of which we use Theorem 1 and the method by K. Kanno [4].

We shall prove this theorem in the section 2.

THEOREM 2. Let $W(x)$ and $V(x)$ be positive functions of $x>0$, such that

$$
(1.9)\quad \begin{cases} \text{(i)} & W(x) \text{ is non-decreasing and unbounded, there exist} \\ & \text{positive constants } H \text{ and } \eta(0<\eta<1) \text{ for which} \\ \text{(ii)} & V(x')/V(x)<H \text{ if } 0<|x'-x|<\eta x, \\ \text{(iii)} & W(x)/V(x)=O(x^{\delta-\alpha+\beta}) \text{ as } x\to\infty \\ & \text{and} \\ \text{(iv)} & x^\alpha W(x)/x^\beta V(x)\to\infty \text{ as } x\to\infty, \\ & \text{where } \delta>0,\ \alpha>-1, \text{ and } \beta \text{ are real numbers.} \end{cases}
$$

And

$$(1.10)\qquad S_n^\delta=o(n^\alpha W(n)) \text{ as } n\to\infty,$$

and

$$(1.11)\qquad s_n=O(n^\beta V(n)) \text{ as } n\to\infty,$$

together imply, for any γ such that $0<\gamma<\delta$,

$$(1.12)\qquad S_n^\gamma=o[n^{\beta(\delta-\gamma)/\delta+\alpha\gamma/\delta}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}] \text{ as } n\to\infty.$$

2. Proof of Theorem 2. By (1.9) (i), $\alpha>-1$ and (1.10),

$$
\begin{aligned}
S_n^{\delta+1}&=\sum_{\nu=0}^n S_\nu^\delta \\
&=o[\sum_{\nu=0}^n \nu^\alpha W(\nu)] \\
&=o(n^{\alpha+1}W(n)).
\end{aligned}
$$

If we put

$$T_n^\delta=\sum_{\nu=0}^n A_{n-\nu}^{\delta-1}\nu s_\nu,$$

we get

$$(2.1) \qquad T_n^\delta=(\delta+n)S_n^\delta-(\delta+1)S_n^{\delta+1} =o(n^{\alpha+1}W(n)).$$

Now $ns_n=O(n^{\beta+1}V(n))$ from (1.11). Suppose $0<\delta\leqq 1$, then the hypotheses of Theorem 1 are satisfied with s_n replaced by ns_n, $W(n)$ by $n^{\alpha+1}W(n)$, $V(n)$ by $n^{\beta+1}V(n)$, respectively.

In fact, $n^{\alpha+1}W(n)$ is non-decreasing and unbounded, and

$$\begin{aligned}(n')^{\beta+1}V(n')/n^{\beta+1}V(n)&=(n'/n)^{\beta+1}V(n')/V(n)\\ &\leqq(\eta+1)^{\beta+1}V(n')/V(n)\\ &\leqq H'V(n')/V(n),\ 0<|n'-n|<\eta n,\end{aligned}$$

where H' is constant, and

$$(n^{\alpha+1}W(n)/n^{\beta+1}V(n))^{1/\delta}=n^{(\alpha-\beta)/\delta}\,(W(n)/V(n))^{1/\delta}=O(n).$$

Therefore, by Theorem 1, for $0<\gamma<\delta$, we have

$$(2.2) \qquad \begin{aligned}T_n^\gamma&=o[(n^{\beta+1}V(n))^{1-\gamma/\delta}\,(n^{\alpha+1}W(n)^{\gamma/\delta}]\\ &=o[n^{(\beta+1)(\delta-\gamma)/\delta+(\alpha+1)\gamma/\delta}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}] \text{ as } n\to\infty.\end{aligned}$$

Since $0<\gamma<\delta\leqq 1$, by (1.10),

$$(2.3) \qquad \begin{aligned}S_n^{\gamma+1}&=\sum_{\nu=0}^n A_{n-\nu}^{\gamma-\delta}S_\nu^\delta\\ &=o[\sum_{\nu=0}^n(n-\nu)^{\gamma-\delta}\nu^\alpha W(\nu)]\\ &=o[n^{\alpha+\gamma-\delta+1}W(n)] \text{ as } n\to\infty.\end{aligned}$$

From (2.1), (2.2), (2.3), and (1.9) (iii),

$$\begin{aligned}S_n^\gamma&=(1/(n+\gamma))\,(T_n^\gamma+(\gamma+1)S_n^{\gamma+1})\\ &=o[n^{(\beta+1)(\delta-\gamma)/\delta+(\alpha+1)\gamma/\delta-1}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}+n^{\alpha+\gamma-\delta}W(n)]\\ &=o[n^{\beta(\delta-\gamma)/\delta+\alpha\gamma/\delta}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}(1+n^{-(\delta-\gamma)(\beta-\alpha+\delta)/\delta}(W(n)/V(n))^{1-\gamma/\delta}]\\ &=o[n^{\beta(\delta-\gamma)/\delta+\alpha\gamma/\delta}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}] \text{ as } n\to\infty,\end{aligned}$$

which is the required result for $0<\delta\leqq 1$.

Next if $1<\delta\leqq 2$, suppose that $0\leqq\gamma<\delta-1$, let us prove the result with γ replaced by $\gamma+1$.

By (2.2) and (1.9) (iii),

$$\begin{aligned}S_n^\gamma&=(1/(n+\gamma))\,[T_n^\gamma+(\gamma+1)S_n^{\gamma+1}]\\ &=o[n^{\beta(\delta-\gamma)/\delta+\alpha\gamma/\delta}(V(n))^{1-\gamma/\delta}(W(n))^{\gamma/\delta}] \text{ as } n\to\infty.\end{aligned}$$

The general case is proved by induction.

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