# 110. On Lacunary Trigonometric Series

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§1. **Introduction.** In [2] R. Salem and A. Zygmund proved the

*Theorem. Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *where* $\{n_k\}$ *is a sequence of positive integers satisfying*

(1.1) $$n_{k+1}>n_k(1+c), \quad \textit{for some } c>0,$$

*and* $\{a_k\}$ *an arbitrary sequence of real numbers for which*

$$A_N\longrightarrow+\infty, \textit{ and } |a_N|=o(A_N), \qquad \textit{as } N\longrightarrow+\infty.$$

*Then we have, for any set* $E\subset[0, 1]$ *of positive measure and* $x$,

(1.2) $$\lim_{N\to\infty}|\{t;\ t\in E,\ S_N(t)\le xA_N\}|/|E|=(2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-u^2/2)du.^{*)}$$

Recently, it is proved that the lacunarity condition (1.1) can be relaxed in some cases (c.f. [1] and [4]). But in [1] it is pointed out that to every constant $c>0$, there exists a sequence $\{n_k\}$ for which $n_{k+1}>n_k(1+ck^{-1/2})$ but (1.2) is not true for $a_k=1$ and $E=[0,1]$.

The purpose of the present note is to prove the following

*Theorem. Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi n_k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *where* $\{n_k\}$ *is a sequence of positive integers satisfying*

(1.3) $$n_{k+1}>n_k(1+ck^{-\alpha}), \quad \textit{for some } c>0 \textit{ and } 0\le\alpha\le 1/2,$$

*and* $\{a_k\}$ *an arbitrary sequence of real numbers for which*

(1.4) $$A_N\longrightarrow+\infty, \textit{ and } |a_N|=o(A_NN^{-\alpha}), \qquad \textit{as } N\longrightarrow+\infty.$$

*Then* (1.2) *holds, for any set* $E\subset[0, 1]$ *of positive measure.*

From the above theorem we can easily obtain the

*Corollary. Under the conditions* (1.3) *and* (1.4), *we have*

(1.5) $$\limsup_{N\to+\infty}\Big|\sum_{k=1}^{N} a_k\cos 2\pi n_k(t+\alpha_k)\Big|=+\infty, \qquad \text{a.e. } \textit{in } t.$$

For the proof of our theorem we use the following lemma which is a special case of Theorem 1 in [3].

*Lemma* 1. *Let* $S_N(t)=\sum_{k=1}^{N} a_k \cos 2\pi k(t+\alpha_k)$ *and* $A_N=(2^{-1}\sum_{k=1}^{N} a_k^2)^{1/2}$, *then we put* $\Delta_k(t)=S_{2^{k+1}}(t)-S_{2^k}(t)$ *and* $B_N=A_{2^{N+1}}$. *Suppose if*

$$B_N\longrightarrow+\infty, \textit{ and } \sup_t|\Delta_N(t)|=o(B_N), \qquad \textit{as } N\longrightarrow+\infty,$$

*and*

$$\lim_{N\to\infty}\int_0^1\Big|B_N^{-2}\sum_{K=1}^{N}\{\Delta_K^2(t)+2\Delta_K(t)\Delta_{K+1}(t)\}-1\Big|dt=0,$$

*then* (1.2) *holds, for any set* $E\subset[0, 1]$ *of positive measure.*

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*) $|E|$ denotes the Lebesgue measure of the measurable set $E$.

§2. **Some Lemmas.** From now on let us assume that $\{n_k\}$ satisfies the gap condition (1.3), and put, for $k=1, 2, \cdots$,

(2.1) $$p(k)=\max\{m;\ n_m\leq 2^k\}.$$

If $p(k)+1<p(k+1)$, (1.3) implies that

$$2>n_{p(k+1)}/n_{p(k)+1}>\prod_{m=p(k)+1}^{p(k+1)-1}(1+cm^{-\alpha})>1+c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1).$$

Thus we have

(2.2) $$\{p(k+1)-p(k)\}=O(p^{\alpha}(k)), \qquad \text{as } k\to+\infty.$$

Next for any given $k$, $j$, $q$, and $h$ satisfying

(2.3) $$p(j)+1<h\leq p(j+1)<p(k)+1<q\leq p(k+1),$$

we consider the number of solutions $(n_r, n_i)$ of the equation:

(2.4) $$n_q-n_r=n_h-n_i, \text{ where } p(j)<i<h \text{ and } p(k)<r<q.$$

Then from the above relations, it is seen that

$$n_r=n_q-(n_h-n_i)>n_q-2^j>n_q(1-2^{j-k})>n_q(1+2^{j-k+1})^{-1}.$$

Therefore, the $n_r'$s in (2.4) must satisfy the inequalities

$$n_q(1+2^{j-k+1})^{-1}<n_r<n_q, \text{ and } p(k)<r<q.$$

Thus if $m_1$(or $m_2$) denotes the smallest (or the largest) index of $n_r'$s satisfying the above inequalities, we have, by (1.3),

$$(1+2^{j-k+1})>n_{m_2+1}/n_{m_1}>\prod_{m=m_1}^{m_2}(1+cm^{-\alpha})>1+c(m_2-m_1+1)p^{-\alpha}(k+1).$$

Since $p(k+1)/p(k)\to 1$, as $k\to+\infty$, the number of $n_r'$s in (2.4) is at most $2^{j-k}Cp^{\alpha}(k)$, for some positive constant $C$. Further for any given $q$, $r$, and $h$, there exists at most one $n_i$ satisfying (2.4). Hence we can obtain the following

*Lemma* 2. *For any given $k$, $j$, $q$, and $h$ satisfying* (2.3), *the number of solutions $(n_r, n_i)$ of* (2.4) *is at most $2^{j-k}Cp^{\alpha}(k)$.*

In the same way we can prove the following

*Lemma* 3. *For any given $k$, $j$, $q$, and $h$ such that $j\leq k-2$, $p(j+1)<h\leq p(j+2)$, and $p(k+1)<q\leq p(k+2)$, the number of solutions $(n_r, n_i)$ of the equation $n_q-n_r=n_h-n_i$ where $p(j)<i\leq p(j+1)$ and $p(k)<r\leq p(k+1)$ is at most $2^{j-k}Cp^{\alpha}(k)$, where $C$ is a positive constant.*

§3. **Proof of the Theorem.** To simplify the computations we will work out the proof of the theorem only for cosine series, the proof of the general case follows the same lines.

I. First let us put, for $k=1, 2, \cdots$,

$$\Delta_k(t)=\sum_{m=p(k)+1}^{p(k+1)} a_m\cos 2\pi n_m t,^{*)} \text{ and } B_k=A_{p(k+1)}.$$

Then we have, by (1.4) and (2.2),

(3.1) $$\sup_t|\Delta_k(t)|\leq\sum_{m=p(k)+1}^{p(k+1)}|a_m|\leq\max_{p(k)<m\leq p(k+1)}|a_m|\{p(k+1)-p(k)\}$$
$$=o(B_kp^{-\alpha}(k)\{p(k+1)-p(k)\})=o(B_k), \qquad \text{as } k\to+\infty.$$

On the other hand we have

(3.2) $$\Delta_k^2(t)-\|\Delta_k\|^2=U_k(t)+V_k(t),^{**)}$$

*) If $p(k+1)=p(k)$, then we put $\Delta_k(t)=0$.

**) For any $f(t)\in L_2(0, 1)$, $\|f\|=\left(\int_0^1 f^2(t)dt\right)^{1/2}$.

where

$$(3.3)\qquad \begin{cases} U_k(t)=2^{-1}\sum_{q=p(k)+1}^{p(k+1)} a_q \sum_{r=p(k)+1}^{p(k+1)} a_r \cos 2\pi(n_q+n_r)t, \\ V_k(t)=\sum_{q=p(k)+2}^{p(k+1)} a_q \sum_{r=p(k)+1}^{q-1} a_r \cos 2\pi(n_q-n_r)t. \end{cases}$$

Then the Minkowski inequality and (3.1) imply that

$\| U_k \| \leq \sum_{q=p(k)+1}^{p(k+1)} | a_q | \, \| \Delta_k \| = o(B_k \| \Delta_k \|)$, and

$$\| V_k \| = o(B_k \| \Delta_k \|), \qquad \text{as } k\to+\infty.$$

Thus we have $\sum_{k=1}^{N} \| V_k \|^2 = o(\sum_{k=1}^{N} B_k^2 \| \Delta_k \|^2) = o(B_N^4)$, as $N\to+\infty$. Further since the sequence of functions $\{U_k(t)\}$ is orthogonal, it is seen that $\| \sum_{k=1}^{N} U_k \|^2 = o(\sum_{k=1}^{N} B_k^2 \| \Delta_k \|^2) = o(B_N^4)$, as $N\to+\infty$. Therefore, we have, by (3.2) and the Schwarz inequality,

$$(3.4)\qquad \| \sum_{k=1}^{N} \{\Delta_k^2(t) - \| \Delta_k \|^2\} \|^2 \leq 2 \, \| \sum_{k=1}^{N} U_k(t) \|^2 + 2 \, \| \sum_{k=1}^{N} V_k(t) \|^2$$
$$= o(B_N^4) + 4 \sum_{k=2}^{N} \sum_{j=1}^{k-1} \int_0^1 V_k(t) V_j(t) dt, \qquad \text{as } N\to+\infty.$$

In the same way it is seen that

$$(3.5)\qquad \| \sum_{k=1}^{N} \Delta_k(t) \Delta_{k+1}(t) \|^2 = o(B_N^4) + \sum_{k=3}^{N} \sum_{j=1}^{k-2} \int_0^1 W_k(t) W_j(t) dt, \qquad \text{as } N\to+\infty,$$

where $W_k(t) = \sum_{q=p(k+1)+1}^{p(k+2)} a_q \sum_{r=p(k)+1}^{p(k+1)} a_r \cos 2\pi(n_q-n_r)t$.

II. If $k>j$, then we have, by Lemma 2 and (3.3),

$$\left| \int_0^1 V_k(t) V_j(t) dt \right|$$
$$\leq 2^{j-k} C p^\alpha(k) \, ( \max_{p(k)<r<p(k+1)} | a_r | ) ( \max_{p(j)<i<p(j+1)} | a_i | ) \sum_{q=p(k)+1}^{p(k+1)} | a_q | \sum_{h=p(j)+1}^{p(j+1)} | a_h |.$$

On the other hand (1.4) and (2.2) imply that $( \max_{p(k)<r<p(k+1)} | a_r | ) = o(B_k p^{-\alpha}(k))$ and $\sum_{q=p(k)+1}^{p(k+1)} | a_q | = O(\| \Delta_k \| \{p(k+1)-p(k)\}^{1/2}) = O(\| \Delta_k \| (p(k))^{\alpha/2})$, as $k\to+\infty$. Thus, we have

$$\sum_{k=2}^{N} \sum_{j=1}^{k-1} \left| \int_0^1 V_k(t) V_j(t) dt \right|$$
$$= o(B_N^2) \sum_{k=2}^{N} \| \Delta_k \| \, p^{\alpha/2}(k) \sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \| \{p(j)+1\}^{-\alpha/2}, \qquad \text{as } N\to+\infty.$$

Further since $p(k+1)/p(k)\to 1$, $p(k)\to+\infty$ and $p(k)=O(k^2)$, as $k\to+\infty$, we can find a constant $C_0$ such that $\sum_{j=1}^{k-1} 2^{j-k}\{p(j)+1\}^{-\alpha} < C_0\{p(k)+1\}^{-\alpha}$, for all $k$. Therefore, we have, by the Schwarz inequality,

$$\sum_{k=2}^{N} \sum_{j=1}^{k-1} \left| \int_0^1 V_k(t) V_j(t) dt \right|$$
$$= o(B_N^2) \sum_{k=2}^{N} \| \Delta_k \| p^{\alpha/2}(k) \{\sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \|^2\}^{1/2} \{\sum_{j=1}^{k-1} 2^{j-k} (p(j)+1)^{-\alpha}\}^{1/2}$$
$$= o(B_N^2) \sum_{k=2}^{N} \| \Delta_k \| (\sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \|^2)^{1/2}$$
$$= o(B_N^2) (\sum_{k=2}^{N} \| \Delta_k \|^2)^{1/2} (\sum_{k=2}^{N} \sum_{j=1}^{k-1} 2^{j-k} \| \Delta_j \|^2)^{1/2}$$

$$=o(B_N^3)(\sum_{j=1}^{N-1}\|\Delta_j\|^2\sum_{k=j+1}^{N}2^{j-k})^{1/2}=o(B_N^4), \qquad \text{as } N\to+\infty.$$

Hence, we can obtain from (3.4)

$$\|\sum_{k=1}^{N}\{\Delta_k^2(t)-\|\Delta_k\|^2\}\|=o(B_N^2), \qquad \text{as } N\to+\infty. \tag{3.6}$$

Using the same arguments, we have, by Lemma 3 and (3.5),

$$\|\sum_{k=1}^{N}\Delta_k(t)\Delta_{k+1}(t)\|=o(B_N^2), \qquad \text{as } N\to+\infty. \tag{3.7}$$

From (3.6) and (3.7) it follows that

$$\|B_N^{-2}\sum_{k=1}^{N}\{\Delta_k^2(t)+2\Delta_k(t)\Delta_{k+1}(t)\}-1\|=o(1), \qquad \text{as } N\to+\infty. \tag{3.8}$$

III. By (3.1), (3.8), and Lemma 1, we can complete the proof.

§ 4. **Proof of the Corollary.** Let us assume that (1.5) does not hold. If we put $S_N(t)=\sum_{k=1}^{N}a_k\cos 2\pi n_k(t+\alpha_k)$, then there exist a set $E\subset[0, 1]$ of positive measure, an integer $N_0$ and a positive number $M$ such that $|S_N(t)|<M$ for $t\in E$ and $N>N_0$. From our theorem it is seen that $|\{t;\, t\in E,\ |S_N(t)|<M\}|\to 0$, as $N\to+\infty$. This is a contradiction.

## References


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