147. *Boolean Elements in Lukasiewicz Algebras. II*

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0. *INTRODUCTION.* R. Cignoli has proved the following result:
0.1. *THEOREM*: *Let A be a Kleene algebra. It is possible to define on A a structure of Lukasiewicz algebra if and only if the family B of all Boolean elements of A has the following properties*:

B 1) *B is separating.*

B 2) *B is lower relatively complete.*

The purpose of this note is to show that if, instead of a Kleene algebra, A is a distributive lattice with first (0) and last element (1), then we can define on A a structure of Lukasiewicz algebra if and only if the family B has the properties B 1), B 2), and

B 3) *B is upper relatively complete.*

We shall use the notations and definitions of [1].

In § 1 we introduce an alternative definition of Lukasiewicz algebra which is useful for the purpose of this paper.

1. *DEFINITION OF LUKASIEWICZ ALGEBRAS.* We can define the notion of (three-valued) Lukasiewicz algebra introduced and developed by Gr. Moisil [3], [4], [5] in the following way [6], [7]:
1.1. *DEFINITION*: *A* (*three-valued*) *Lukasiewicz algebra is a system $(A, 1, \wedge, \vee, \sim, \nabla)$ where $(A, 1, \wedge, \vee, \sim)$ is a de Morgan lattice and ∇ is a unary operator defined on A satisfying the following axioms*:

L 1) $\sim x \vee \nabla x = 1$, L 2) $x \wedge \sim x = \sim x \wedge \nabla x$,

L 3) $\nabla(x \wedge y) = \nabla x \wedge \nabla y$.

In [6] (Theorem 4.3) it was proved that in a (three-valued) Lukasiewicz algebra the operation $\sim$ also satisfies the condition

K) $$x \wedge \sim x \leq y \vee \sim y,$$

that is, the system $(A, 1, \wedge, \vee, \sim)$ is not only a de Morgan algebra but a Kleene algebra.

A. Monterio has proved that if we postulate the condition K), then we can replace axiom L 3) of definition 1.1 by the weaker

$L'3)$ $$\nabla(x \wedge y) \leq \nabla x \wedge \nabla y.$$

More exactly:
1.2. *THEOREM*: *Let $(A, 1, \wedge, \vee, \sim, \nabla)$ be a system such that $(A, 1, \wedge, \vee, \sim)$ is a Kleene algebra and ∇ is a unary operator defined on A satisfying axioms L 1), L 2), and $L'3)$. Then* $(A, 1, \wedge,$

$\vee, \sim, \nabla)$ *is a (three-valued) Lukasiewicz algebra.*

PROOF: As Kleene algebras are special kind of de Morgan algebras, to prove the theorem we need show that

(1) $$\nabla x \wedge \nabla y \leq \nabla (x \wedge y).$$

We will prove (1) in the following steps:

a) $x \leq \nabla x$.

By $L\,1)$ we have

$$x \wedge (\sim x \vee \nabla x) = x \wedge 1 = x,$$

then

$$(x \wedge \sim x) \vee (x \wedge \nabla x) = x$$

and, recalling $L\,2)$, we can write:

$$(\sim x \wedge \nabla x) \vee (x \wedge \nabla x) = x.$$

Therefore

$$x = (\sim x \vee x) \wedge \nabla x \leq \nabla x.$$

b) If $\sim x \wedge z \leq x$, then $z \leq \nabla x$.

Suppose that $\sim x \wedge z \leq x$, we have

$$(\sim x \wedge z) \vee \nabla x \leq x \vee \nabla x$$

and then, by a), we can write:

$$(\sim x \vee \nabla x) \leq (z \vee \nabla x) \leq \nabla x$$

and recalling $L\,1)$

$$z \vee \nabla x \leq \nabla x,$$

therefore

$$z \leq \nabla x.$$

c) $\sim x \wedge \nabla x \wedge \nabla y \leq x$.

Using $L\,2)$ we can write:

$$\sim x \wedge \nabla x \wedge \nabla y = \sim x \wedge x \wedge \nabla y \leq x.$$

d) $\sim x \wedge \nabla x \wedge \nabla y \leq y$.

By $L\,2)$, $K)$, and a) we have

$$\sim x \wedge \nabla x \wedge \nabla y = \sim x \wedge x \wedge \nabla y \leq (\sim y \vee y) \wedge \nabla y = (\sim y \wedge \nabla y) \vee (y \wedge \nabla y)$$
$$= (\sim y \wedge \nabla y) \vee y = (y \wedge \sim y) \vee y = y.$$

From c) and d) we have

e) $\sim x \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

From e), interchanging x by y, we have

f) $\sim y \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

From e) and f), taking acount of $M\,2)$ it follows that

g) $\sim (x \wedge y) \wedge \nabla x \wedge \nabla y \leq x \wedge y$.

Finally, from b) and g) we have (2).

2. *CHARACTERISTIC PROPERTIES OF BOOLEAN ELEMENTS OF LUKASIEWICZ ALGEBRAS.* Let $(A, 0, 1, \wedge, \vee)$ be a distributive lattice with first and last element. If $x \in A$ has a Boolean complement, we shall denote it by $-x$. It is convenient to recall the following property:

2.1. *If z is a Boolean element of A, then for all $x \in A$*

$$x \wedge z = 0 \text{ is equivalent to } x \leq -z.$$

2.2. *LEMMA: Let $(A, 0, 1, \wedge, \vee)$ be a distributive lattice and let B be the sublattice of all Boolean elements of A.*

a) *If B is lower relatively complete, then the operator ∇ defined on A by the formula:*

$$\nabla x = \wedge \{b \in B : x \leq b\}$$

has the following properties:

$C\,1)$ $\nabla 0 = 0$, $C\,2)$ $x \leq \nabla x$, $C\,3)$ $\nabla(x \vee y) = \nabla x \vee \nabla y$,
$C\,4)$ $\nabla\nabla x = \nabla x$. $C\,5)$ If $x \leq y$, then $\nabla x \leq \nabla y$,
$C\,6)$ $\nabla x = x$ if and only if $x \in B$, $C\,7)$ $\nabla(x \wedge \nabla y) = \nabla x \wedge \nabla y$.

b) *If B is upper relatively complete, then the operator Δ defined on A by the formula:*

$$\Delta x = \vee \{b \in B : b \leq x\}$$

has the following properties:

$I\,1)$ $\Delta 1 = 1$, $I\,2)$ $\Delta x \leq x$, $I\,3)$ $\Delta(x \wedge y) = \Delta x \wedge \Delta y$,
$I\,4)$ $\Delta\Delta x = \Delta x$, $I\,5)$ If $x \leq y$, then $x\Delta \leq \Delta y$,
$I\,6)$ $\Delta x = x$ if and only if $x \in B$, $I\,7)$ $\Delta(x \vee \Delta y) = \Delta x \vee \Delta y$.

PROOF: a) The properties $C\,1)$–$C\,6)$ are a consequence of the fact that B is a sublattice of A containing 0 and 1 and lower relatively complete (see [1]).

Let us prove $C\,7)$:

As $x \wedge \nabla y \leq x$, it follows from $C\,5)$ that

(1) $$\nabla(x \wedge \nabla y) \leq \nabla x$$

and since $x \wedge \nabla y \leq \nabla y$, from $C\,5)$ and $C\,4)$ we have

(2) $$\nabla(x \wedge \nabla y) \leq \nabla y.$$

On the other hand, by $C\,2)$ we can write:

$$x \wedge \nabla x \wedge \nabla y \wedge -\nabla(x \wedge \nabla y) = (x \wedge \nabla y) \wedge -\nabla(x \wedge \nabla y)$$
$$\leq (x \wedge \nabla y) \wedge -\nabla(x \wedge \nabla y) = 0.$$

Since $\nabla x \wedge \nabla y \wedge -\nabla(x \wedge \nabla y) \in B$, we have (by 2.1)

(3) $$x \leq -\nabla x \vee -\nabla y \vee \nabla(x \wedge \nabla y).$$

Meeting both sides of (3) with ∇x and using $C\,2)$ and (1) we have

$$x = x \wedge \nabla x \leq (\nabla x \wedge -\nabla y) \wedge \nabla(x \wedge \nabla y) \leq \nabla x$$

Hence, by $C\,5)$, $C\,3)$, and $C\,6)$ it follows that

$$\nabla x = (\nabla x \wedge -\nabla y) \vee \nabla(x \wedge \nabla y)$$

and then, by (2)

$$\nabla x \wedge \nabla y = \nabla(x \wedge \nabla y).$$

It is not necessary to prove b), for it is the dual form of a).

Q.E.D.

(Compare this result with [2]).

We shall say that a sublattice B of a lattice A is *relatively complete* if it is both lower and upper relatively complete.

2.3. *THEOREM*: *Let* $(A, 0, 1, \wedge, \vee)$ *be a distributive lattice with first and last element such that the sublattice* B *of all its Boolean elements is relatively complete and separating. Then, defining the operators* ∇, Δ, *and* $\sim$ *by the formulae*:

$$\nabla x = \wedge\{b \in B : x \leq b\}, \qquad \Delta x = \vee\{b \in B : b \leq x\},$$

$$\sim x = (-\Delta x \wedge x) \vee -\nabla x,$$

the system $(A, 1, \wedge, \vee, \sim, \nabla)$ *is a (three-valued) Lukasiewicz algebra.*

PROOF: We shall use all properties shown in 2.2 without reference. The theorem will be proved in the following steps:

a) $\nabla(x \wedge y) \leq \nabla x \wedge \nabla y$.

It follows immediately from $C\,5)$.

b) $\sim x \vee \nabla x = 1$.

It easily follows from the definition of $\sim x$.

c) $x \wedge \sim x = \sim x \wedge \nabla x$.

Taking account of 2.1, we have $x \wedge -\nabla x = 0$, then

$$x \wedge \sim x = x \wedge ((-\Delta x \wedge x) \vee -\nabla x) = -\Delta x \wedge x.$$

But we also have

$$\sim x \wedge \nabla x = ((-\Delta x \wedge x) \vee -\nabla x) \wedge \nabla x = -\Delta x \wedge x.$$

d) If $z \in B$, then $\sim z = -z$.

By $z \in B$, we have $\Delta z = z = \nabla z$, then

$$\sim z = (-\Delta z \wedge z) \vee -\nabla z = (-z \wedge z) \vee -z = -z.$$

e) $-\Delta x = \sim \Delta x$ and $-\nabla x = \sim \nabla x$.

It is an immediate consequence of d).

f) $\Delta x = \sim \nabla \sim x$.

First of all, we have $-\Delta x = \nabla - \Delta x$, $-\nabla x = \nabla - \nabla x$, hence we can write

$$\begin{aligned}\nabla \sim x &= \nabla((-\Delta x \wedge x) \vee -\nabla x) = \nabla(-\Delta x \wedge x) \vee \nabla - \nabla x \\ &= \nabla(\nabla - \Delta x \wedge x) \vee -\nabla x = (\nabla - \Delta x \wedge \nabla x) \vee -\nabla x \\ &= (-\Delta x \wedge \nabla x) \vee -\nabla x = -\Delta x \vee -\nabla x = -\Delta x\end{aligned}$$

and then f) follows from e).

g) $\nabla x = \sim \Delta \sim x$.

The proof of g) is analogous to that of f).

h) $\sim\sim x = x$.

By e), f), and g) we have

$$\begin{aligned}\sim\sim x &= (-\Delta \sim x \wedge \sim x) \vee -\nabla \sim x = (\nabla x \wedge \sim x) \vee \Delta x \\ &= (\nabla x \wedge ((-\Delta x \wedge x) \vee -\nabla x))) \vee \Delta x = (-\Delta x \wedge x) \vee \Delta x = x.\end{aligned}$$

i) $x \leq y$ if and only if $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

If $x \leq y$, then $\Delta x \leq \Delta y$ and $\nabla x \leq \nabla y$.

Conversely, if $\Delta x \leq \Delta y$, then for all $z' \in B$ we have:

$$z' \leq x \text{ implies } z' \leq y$$

and if $\nabla x \leq \nabla y$, then for all $z \in B$ we have:

$$y \leq z \text{ implies } x \leq z,$$

therefore, by the separating property of B, we must have $x \leq y$.

j) If $x \leq y$, then $\sim y \leq \sim x$.

According to i), it is sufficient to prove that $\Delta \sim x \leq \Delta \sim y$ and $\nabla \sim x \leq \nabla \sim y$.

But by e), g), and h) we have $\Delta \sim y = -\nabla y$ and $\Delta \sim x = -\nabla x$, hence, if $x \leq y$, it follows that $\Delta \sim y \leq \Delta \sim x$. Analogously we can prove $\nabla \sim y \leq \nabla \sim x$.

k) $\sim (x \wedge y) = \sim x \vee \sim y$.

It easily follows from h) and k).

l) $x \wedge \sim x \leq y \vee \sim y$.

As we have shown in the proof of c), $x \wedge \sim x = -\Delta x \wedge x$, thus $\Delta(x \wedge \sim x) = 0$ and a fortiori

(1) $$\Delta(x \wedge \sim x) \leq \Delta(y \vee \sim y).$$

On the other hand, $y \vee \sim y = y \vee (-\Delta y \wedge y) \vee -\nabla y = y \vee -\nabla y$, therefore $\nabla(y \vee \sim y) = 1$, and then we have

(2) $$\nabla(x \wedge \sim x) \leq \nabla(y \vee \sim y)$$

and $x \wedge \sim x \leq y \vee \sim y$ follows from i), (1), and (2).

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