164. On the Absolute Cesàro Summability Factors of Fourier Series*)

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1.1. **Definition.** Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series and s_n^α be the n-th Cesàro mean of order α of the sequence $\{s_n\}$, where s_n is the partial sum of the given series. We say that the series $\sum_{n=0}^{\infty} a_n$ is absolutely summable (C, α), or summable $|C, \alpha|$, if the series $\sum_{n=1}^{\infty} |s_n^\alpha - s_{n-1}^\alpha|$ converges.

A sequence $\{\lambda_n\}$ is said to be convex when $\Delta^2\lambda_n \geqq 0$ $(n=1, 2, \cdots)$, where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, $\Delta^2\lambda_n = \Delta(\Delta\lambda_n)$. It is known[1] that if $\{\lambda_n\}$ is a convex sequence and the series $\sum_{n=1}^{\infty} n^{-1}\lambda_n$ converges, then λ_n is non-negative and non-increasing.

1.2. Let $f(t)$ be a periodic function with period 2π, and integrable in the sense of Lebesgue over $(-\pi, \pi)$. Without any loss of generality we may assume that the constant term in the Fourier series of $f(t)$ is zero, so that

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t),$$

where $A_n(x) = a_n \cos nx + b_n \sin nx$. Let us put

$$s_n(x) = \sum_{v=1}^{\infty} A_v(x), \quad D_n(t) = \frac{1}{2} + \sum_{v=1}^{n} \cos nt = \frac{\sin\left(n+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}$$

and $\phi(t) = \phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$.

1.3. Recently, Pati has proved the following result:

Theorem A.[2] If $\{\lambda_n\}$ is a convex sequence such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n(\log n)^{\frac{1}{2}} < \infty$, then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t=x$ at which

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1) H. C. Chow [1], Lemma 4.

2) T. Pati [2], Theorem 2.

$$\int_0^t |\phi(u)|\,du = o(t). \tag{1.1}$$

2. The object of this paper is to prove the following two theorems:

Theorem 1. If $\{\lambda_n\}$ is a convex sequence such that

$$\sum_{n=1}^{\infty} \frac{\lambda_n (\log n)^{\frac{1}{2}(1-\alpha)}}{n} < \infty \quad (0 \leqq \alpha < 1), \tag{2.1}$$

then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t=x$ at which

$$\Phi(t) = \int_0^t |\phi(u)|\,du = o\left(\frac{t}{\left(\log \frac{1}{t}\right)^{\alpha}}\right), \quad \text{as} \quad t \to 0. \tag{2.2}$$

In the case $\alpha=0$, we get Theorem A. In the limiting case, $\alpha=1$, we have the following

Theorem 2. If $\{\lambda_n\}$ is a convex sequence such that

$$\sum_{n=1}^{\infty} \frac{\lambda_n (\log \log n)^{\frac{1}{2}}}{n} < \infty, \tag{2.3}$$

then $\sum_{n=1}^{\infty} \lambda_n A_n(t)$ is summable $|C, 1|$ at every point $t=x$ at which

$$\Phi(t) = \int_0^t |\phi(u)|\,du = o\left(\frac{t}{\log \frac{1}{t}}\right), \quad \text{as} \quad t \to 0. \tag{2.4}$$

3. **Proof of Theorem 1.** We require the following lemmas:

Lemma 1.[3] If $\{\lambda_n\}$ is a convex sequence such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n < \infty$ and (1.1) holds, then $\sum_{n=1}^{\infty} \lambda_n A_n(x)$ is summable $|C, 1|$ if and only if $\sum_{n=1}^{\infty} n^{-1}\lambda_n |s_n(x) - f(x)| < \infty$.

Lemma 2. If (2.2) holds, then

$$\sum_{v=1}^{n} (s_v(x) - f(x))^2 = o(n(\log n)^{1-\alpha}),$$

and further, by Cauchy's inequality, we have

$$\sum_{v=1}^{n} |s_v(x) - f(x)| = o(n(\log n)^{\frac{1}{2}(1-\alpha)}), \quad \text{as} \quad n \to \infty.$$

Proof. First, under the condition (2.2), we shall estimate the order of the integrals $\int_{\frac{\pi}{n}}^{\pi} \frac{|\phi(t)|}{t}\,dt$ and $\int_{\frac{\pi}{n}}^{\pi} \frac{|\phi(t)|}{t^2}\,dt$. By integration by parts, we get

$$\int_{\frac{\pi}{n}}^{\pi} \frac{|\phi(t)|}{t}\,dt = \left[\frac{\Phi(t)}{t}\right]_{\frac{\pi}{n}}^{\pi} + \int_{\frac{\pi}{n}}^{\pi} \frac{\Phi(t)}{t^2}\,dt \tag{3.1}$$

3) T. Pati [2], Theorem 1.

$$=O(1)+o\left(\frac{1}{(\log n)^{\alpha}}\right)+o\left(\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t\left(\log\frac{1}{t}\right)^{\alpha}}dt\right)=o((\log n)^{1-\alpha}).$$

$$(3.2)\qquad \int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t^2}dt=\left[\frac{\Phi(t)}{t^2}\right]_{\frac{\pi}{n}}^{\pi}+2\int_{\frac{\pi}{n}}^{\pi}\frac{\Phi(t)}{t^3}dt$$

$$=O(1)+o\left(\frac{n}{(\log n)^{\alpha}}\right)+o\left(\int_{\frac{\pi}{n}}^{\pi}\frac{1}{t^2\left(\log\frac{1}{t}\right)^{\alpha}}dt\right)=o\left(\frac{n}{(\log n)^{\alpha}}\right).$$

Now

$$\sum_{v=1}^{n}(s_v(x)-f(x))^2=\sum_{v=1}^{n}\left(\frac{2}{\pi}\int_0^{\pi}\phi(t)\frac{\sin vt}{t}dt+o(1)\right)^2$$

$$=\sum_{v=1}^{n}\left\{\frac{4}{\pi^2}\int_0^{\pi}\phi(t)\frac{\sin vt}{t}dt\int_0^{\pi}\phi(u)\frac{\sin vu}{u}du+o\left(\int_0^{\pi}\phi(t)\frac{\sin vt}{t}dt\right)+o(1)\right\}$$

$$=\frac{4}{\pi^2}\int_0^{\pi}\frac{\phi(t)}{t}dt\int_0^{\pi}\frac{\phi(u)}{u}\left(\sum_{v=1}^{n}\sin vt\sin vu\right)du+o\left(\sum_{v=1}^{n}\int_0^{\pi}\phi(t)\frac{\sin vt}{t}dt\right)+o(n)$$

$$=I_1+o(\sqrt{I_1})+o(n),$$

where

$$I_1=\frac{4}{\pi^2}\int_0^{\pi}\frac{\phi(t)}{t}dt\int_0^{\pi}\frac{\phi(u)}{u}\left(\sum_{v=1}^{n}\sin vt\sin vu\right)du.$$

We shall devide I_1 into four parts

$$I_1=\frac{4}{\pi^2}\left(\int_0^{\frac{\pi}{n}}\int_0^{\frac{\pi}{n}}+\int_0^{\frac{\pi}{n}}\int_{\frac{\pi}{n}}^{\pi}+\int_{\frac{\pi}{n}}^{\pi}\int_0^{\frac{\pi}{n}}+\int_{\frac{\pi}{n}}^{\pi}\int_{\frac{\pi}{n}}^{\pi}\right)=J_1+J_2+J_3+J_4.$$

By condition (2.2), we get

$$|J_1|\leqq\frac{4}{\pi^2}\int_0^{\frac{\pi}{n}}|\phi(t)|dt\int_0^{\frac{\pi}{n}}|\phi(u)|\left(\sum_{v=1}^{n}v^2\right)du=o\left(\frac{n}{(\log n)^{2\alpha}}\right).$$

By (2.2) and (3.1), we get

$$|J_2|\leqq\frac{4}{\pi^2}\int_0^{\frac{\pi}{n}}|\phi(t)|dt\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(u)|}{u}\left(\sum_{v=1}^{n}v\right)du=o\left(\frac{1}{n(\log n)^{\alpha}}(\log n)^{1-\alpha}n^2\right)$$

$$=o\left(\frac{n(\log n)^{1-\alpha}}{(\log n)^{\alpha}}\right).$$

J_3 is equal to J_2. Hence it remains to estimate J_4:

$$J_4=\frac{2}{\pi^2}\int_{\frac{\pi}{n}}^{\pi}\frac{\phi(t)}{t}dt\int_{\frac{\pi}{n}}^{\pi}\frac{\phi(u)}{u}\left(\sum_{v=1}^{n}(\cos v(u-t)-\cos v(u+t))\right)du$$

$$=\frac{2}{\pi^2}\int_{\frac{\pi}{n}}^{\pi}\frac{\phi(t)}{t}dt\int_{\frac{\pi}{n}}^{\pi}\frac{\phi(u)}{u}(D_n(u-t)-D_n(u+t))du$$

$$=o\left(\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(u)|}{u}\frac{\left|\sin\left(n+\frac{1}{2}\right)(u-t)\right|}{|u-t|}du\right)+$$

$$+o\left(\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(u)|}{u}\frac{\left|\sin\left(n+\frac{1}{2}\right)(u+t)\right|}{u+t}du\right)=O(J_4'+J_4'').$$

Then

$$J_4'=\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\left\{\left(\int_{|u-t|\leqq\frac{\pi}{2n}}+\int_{|u-t|>\frac{\pi}{2n}}\right)\frac{|\phi(u)|}{u}\frac{\left|\sin\left(n+\frac{1}{2}\right)(u-t)\right|}{|u-t|}du\right\}$$
$$=J_{41}'+J_{42}'.$$

By integration by parts and by (2.2) and (3.1), we get

$$J_{41}'\leqq\left(n+\frac{1}{2}\right)\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\int_{t-\frac{\pi}{2n}}^{t+\frac{\pi}{2n}}\frac{|\phi(u)|}{u}du$$

$$=\left(n+\frac{1}{2}\right)\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\left\{\left[\frac{\Phi\left(t+\frac{\pi}{2n}\right)}{t+\frac{\pi}{2n}}-\frac{\Phi\left(t-\frac{\pi}{2n}\right)}{t-\frac{\pi}{2n}}\right]+\int_{t-\frac{\pi}{2n}}^{t+\frac{\pi}{2n}}\frac{\Phi(u)}{u^2}du\right\}$$

$$=o\left(n\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}\left\{\left(\log\left(t+\frac{\pi}{2n}\right)\right)^{-\alpha}-\left(\log\left(t-\frac{\pi}{2n}\right)\right)^{-\alpha}\right\}dt\right)+$$

$$+o\left(n\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}\left\{\left(\log\left(t+\frac{\pi}{2n}\right)\right)^{1-\alpha}-\left(\log\left(t-\frac{\pi}{2n}\right)\right)^{1-\alpha}\right\}dt\right)$$

$$=o(n(\log n)^{1-\alpha}),$$

further,

$$J_{42}'\leqq\left(\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{2n}}\int_{t+\frac{\pi}{2n}}^{\pi}+\int_{\frac{\pi}{n}+\frac{\pi}{2n}}^{\pi}\int_{\frac{\pi}{n}}^{t-\frac{\pi}{2n}}\right)\frac{|\phi(t)|}{t}\frac{|\phi(u)|}{u|u-t|}dt\,du=J_{421}'+J_{422}'.$$

By integration by parts and by (2.2) and (3.1), we get

$$J_{421}'=\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{2n}}\frac{|\phi(t)|}{t}dt\left\{\left[\frac{\Phi(u)}{u|u-t|}\right]_{t+\frac{\pi}{2n}}^{\pi}-\int_{t+\frac{\pi}{2n}}^{\pi}\frac{\Phi(u)}{u^2(u-t)^2}(2u-t)du\right\}$$

$$=\frac{\Phi(\pi)}{\pi}\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{2n}}\frac{|\phi(t)|}{t(\pi-t)}dt+o\left(n\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{2n}}\frac{\phi(t)}{t\left(\log\frac{1}{t+\frac{\pi}{2n}}\right)^{\alpha}}dt\right)$$

$$+o\left(\int_{\frac{\pi}{n}}^{\pi-\frac{\pi}{2n}}\frac{|\phi(t)|}{t}dt\left(\int_{t+\frac{\pi}{2n}}^{\pi}\frac{2u-t}{u(u-t)^2\left(\log\frac{1}{u}\right)^{\alpha}}du\right)\right)=o(n(\log n)^{1-\alpha}).$$

J_{422}' is equal to J_{421}'. By (3.1) and (3.2)

$$J_4''\leqq\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(t)|}{t}dt\int_{\frac{\pi}{n}}^{\pi}\frac{|\phi(u)|}{u^2}du=o\left((\log n)^{1-\alpha}\frac{n}{(\log n)^{\alpha}}\right).$$

Thus we get the conclusion

$$\sum_{v=1}^{n}(s_v(x)-f(x))^2=o(n(\log n)^{1-\alpha}).$$

Lemma 3.[4] If $\{\lambda_n\}$ is a convex sequence such that $\sum_{n=1}^{\infty} n^{-1}\lambda_n < \infty$, then

$$\sum_{n=1}^{m} \log(n+1)\Delta\lambda_n = O(1),$$

as $m \to \infty$, and $\lambda_m \log m = o(1)$, as $m \to \infty$.

We shall now proceed to prove Theorem 1.

By Lemma 1, it is enough to prove that

$$\sum_{n=1}^{\infty} n^{-1}\lambda_n \,|\, s_n(x) - f(x) \,| < \infty.$$

By Abel's transformation,[5]

$$\begin{aligned}
&\sum_{v=1}^{n} v^{-1}\lambda_v \,|\, s_v(x) - f(x) \,| \\
&\quad = n^{-1}\lambda_n \sum_{v=1}^{n} |\, s_v(x) - f(x) \,| + \sum_{v=1}^{n-1} \Delta(v^{-1}\lambda_v) \sum_{\mu=1}^{v} |\, s_\mu(x) - f(x) \,| \\
&\quad = n^{-1}\lambda_n \sum_{v=1}^{n} |\, s_v(x) - f(x) \,| + \sum_{v=1}^{n-1} \frac{\lambda_v}{v(v+1)} \sum_{\mu=1}^{v} |\, s_\mu(x) - f(x) \,| + \\
&\quad + \sum_{v=1}^{n-1} \frac{\Delta\lambda_v}{v+1} \sum_{\mu=1}^{v} |\, s_\mu(x) - f(x) \,| = o(\lambda_n (\log n)^{\frac{1}{2}(1-\alpha)}) \\
&\quad + o\Big(\sum_{v=1}^{n-1} v^{-1}\lambda_v (\log v)^{\frac{1}{2}(1-\alpha)}\Big) + o\Big(\sum_{v=1}^{n-1} \Delta\lambda_v (\log v)^{\frac{1}{2}(1-\alpha)}\Big),
\end{aligned}$$

by Lemma 2, and then, by our hypothesis and Lemma 3,

$$\sum_{v=1}^{n} v^{-1}\lambda_v \,|\, s_v(x) - f(x) \,| = o(1),$$

as $n \to \infty$.

4. Proof of Theorem 2. For the proof of Theorem 2, we need Lemma 1, Lemma 3, and the following

Lemma 4. If (2.4) holds, then

$$\sum_{v=1}^{n} (s_v(x) - f(x))^2 = o(n \log\log n),$$

and further, by Cauchy's inequality, we have

$$\sum_{v=1}^{n} |\, s_v(x) - f(x) \,| = o(n (\log\log n)^{\frac{1}{2}}), \quad \text{as} \quad n \to \infty.$$

This lemma can be proved by the same idea as in the proof of Lemma 2.

We shall now prove Theorem 2. By Lemma 1, it is sufficient to prove that

$$\sum_{n=1}^{\infty} n^{-1}\lambda_n \,|\, s_n(x) - f(x) \,| < \infty.$$

By Abel's transformation, we have

$$\sum_{v=1}^{n} v^{-1}\lambda_v \,|\, s_v(x) - f(x) \,|$$

4) T. Pati [2], Lemma 3.

5) Cf. Pati [2].

$$=\sum_{v=1}^{n-1}\Delta(v^{-1}\lambda_v)\sum_{\mu=1}^{v}|s_\mu(x)-f(x)|+n^{-1}\lambda_n\sum_{v=1}^{n}|s_v(x)-f(x)|$$
$$=\sum_{v=1}^{n-1}\frac{\lambda_v}{v(v+1)}\sum_{\mu=1}^{v}|s_\mu(x)-f(x)|+\sum_{v=1}^{n=1}\frac{\Delta\lambda_v}{v+1}\sum_{\mu=1}^{v}|s_\mu(x)-f(x)|$$
$$+n^{-1}\lambda_n\sum_{v=1}^{n}|s_v(x)-f(x)|=o\Big(\sum_{v=1}^{n-1}\frac{\lambda_v}{v}(\log\log v)^{\frac{1}{2}}\Big)+o\Big(\sum_{v=1}^{n-1}\Delta\lambda_v(\log\log v)^{\frac{1}{2}}\Big)$$
$$+o(\lambda_n(\log\log n)^{\frac{1}{2}})\quad\text{(by lemma 4)}$$
$$=o(1),$$

as $n\to\infty$, by our hypothesis and Lemma 3.

References

[1] H. C. Chow: On the summability factors of Fourier Series. Jour. of London Math. Soc. (3), **16**, 215-220 (1941).

[2] T. Pati: On an unsolved problem in the theory of absolute summability factors of Fourier series. Mathematische Zeitschrift, **82**, 106-114 (1963).