198. On Axiom Systems of Propositional Calculi. XIII

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We know some single axioms of the classical propositional calculus. In this note, we shall show that Lukasiewicz-Tarski single axiom of the propositional calculus (see [2]) is equivalent to some axiom systems, for example, (L_3)-system. In their paper, J. Lukasiewicz and A. Tarski do not give the proof of equivalences. For notations and rules of inferences, see [3].

The fundamental Lukasiewicz-Tarski axiom is the following single thesis:

1 *CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv.*

First, we shall prove that the Lukasiewicz-Tarski axiom implies (L_3)-system. The proof is not so easy. Therefore, for the proofs of theses 2 and 4, we shall write the results of substitutions.

1 *p/CqCrq, q/CCNrCsNtCCrCsuCCtsCtu, v/CqCrq, w/p* **C*1 *p/q, q/r, v/CqCrq, w/CCNrCsNtCCrCsu CCtsCtu*—2,
CCCCqCrqCCCNrCsNtCCrCsuCCtsCtuCqCrqCCCNr CsNtCCrCsuCCtsCtuCqCrqCpCqCrq,

2 *CpCqCrq.*

2 *p/CpCqCrq, q/p, r/q* **C*2—3,

3 *CpCqp.*

1 *v/CCpCqpCCNrCsNtCCrCsuCCtsCtu, w/CpCqp* **C*2 *p/CpCqp, q/CCNrCsNtCCrCsuCCtsCtu, r/CpCqp*—*C*3 —*C*3—4,
CCCpCqpCCCNrCsNtCCrCsuCCtsCtuCCpCqpCCNr CsNtCCrCsuCCtsCtuCCpCqpCCpCqpCCNrCsNtCCr CsuCCtsCtu,

4 *CCNrCsNtCCrCsuCCtsCtu.*

4 *r/p, s/q, t/p, u/r* **C*3 *p/Np*—5,

5 *CCpCqrCCpqCpr.*

5 *r/p, q/Cqp* **C*3 *q/Cqp*—*C*3—6,

6 *Cpp.*

3 *p/CCpCqrCCpqCpr, q/Cqr* **C*5—7,

7 *CCqrCCpCqrCCpqCpr*

5 *p/Cqr, q/CpCqr, r/CCpqCpr* **C*7—*C*3 *p/Cqr, q/p*—8,

8 *CCqrCCpqCpr.*

5 *p/Cqr, q/Cpq, r/Cpr* **C*8—9,

9 *CCCqrCpqCCqrCpr.*

3 p/CCCqrCpqCCqrCpr, q/Cpq *C9—10,

10 CCpqCCCqrCpqCCqrCpr.

5 p/Cpq, q/CCqrCpq, r/CCqrCpr *C10—C3 p/Cpq, q/Cqr—11,

11 CCpqCCqrCpr.

3 p/CqCpq, q/CCpqCpr *C3 p/q, q/p—12,

12 CCCpqCprCqCpq.

5 p/CCpqCpr, q/CqCpq, r/CqCpr *C8 p/q, q/Cpq, r/Cpr—C12—13,

13 CCCpqCprCqCpr.

13 p/CpCqr, q/CCpqCpr, r/CqCpr *C13—C5—14,

14 CCpCqrCqCpr.

11 p/q, q/Cpq *C3 p/q, q/p—15,

15 CCCpqrCqr.

14 p/CNpCsNq, q/CpCsp, r/CCqsCqp *C4 r/p, t/q, u/p—C3 q/s—16,

16 CCNpCsNqCCqsCqp.

11 p/CsCNpNq, q/CNpCsNq, r/CCqsCqp *C14 p/s, q/Np, r/Nq, —C16—17,

17 CCsCNpNqCCqsCqp.

14 p/CqCNpNq, q/Cqq, r/Cqp *C17 s/q—18,

18 CCqCNpNqCqp.

15 p/q, q/CNpNq, r/Cqp *C18—19,

19 CCNpNqCqp.

Theses 3, 5, 19, are (L_3)-axiom system.

It has been proved by Y. Arai, one of my colleagues, that (L_3)-system implies (L_1), (L_2), (H), (F), (S_1), (S_2), and (M) (see [4]). Hence, we have that the Lukasiewicz-Tarski axiom implies (L_1), (L_2), (H), (F), (S_1), (S_2), and (M).

Next we shall prove that Lukasiewicz first axiom system (L_1) of propositional calculus implies Lukasiewicz-Tarski axiom:

CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv.

Lukasiewicz has proved that the (L_1)-system implies the following theses (for the detail, see [1]).

1′	Cpp	(16),
2′	CNCpqp	(66),
3′	CCpqCCqrCpr	(1),
4′	CCqrCCpqCpr	(22),
5′	CCNpNqCqp	(49),
6′	CCpCqrCqCpr	(21),
7′	CCpqCNqNp	(46),
8′	CCpCqrCCpqCpr	(35),

9′ *CpCqp* (18),

10′ *CCNpqCCqpp* (15).

The numbers in brackets represent the number of theses in Elements of Mathematical Logic by Lukasiewicz (see [1]).

Assuming these theses, we shall prove the Lukasiewicz-Tarski thesis.

3′ *p*/*NCsu*, *q*/*s*, *r*/*Ctr* **C*2′ *p*/*s*, *q*/*u*—1,

1 *CCsCtrCNCsuCtr*.

4′ *q*/*CNrNt*, *r*/*Ctr*, *p*/*s* **C*5′ *p*/*r*, *q*/*t*—2,

2 *CCsCNrNtCsCtr*.

3′ *p*/*CsCNrNt*, *q*/*CsCtr*, *r*/*CNCsuCtr* **C*2—*C*1—3,

3 *CCsCNrNtCNCsuCtr*.

3′ *p*/*CNrCsNt*, *q*/*CsCNrNt*, *r*/*CNCsuCtr* **C*6′ *p*/*Nr*, *q*/*s*, *r*/*Nt*—*C*3—4,

4 *CCNrCsNtCNCsuCtr*.

4′ *q*/*Ctr*, *r*/*CNrNt*, *p*/*NCsu* **C*7′ *p*/*t*, *q*/*r*—5,

5 *CCNCsuCtrCNCsuCNrNt*.

3′ *p*/*CNCsuCtr*, *q*/*CNCsuCNrNt*, *r*/*CCNCsuNrCNCsuNt* **C*5—*C*8′ *p*/*NCsu*, *q*/*Nr*, *r*/*Nt*—6,

6 *CCNCsuCtrCCNCsuNrCNCsuNt*.

3′ *p*/*Cpq*, *q*/*CNqNp*, *r*/*CNqNr* **C*7′—7,

7 *CCCNqNpCNqNrCCpqCNqNr*.

3′ *p*/*CCNqNpCNqNr*, *q*/*CCpqCNqNr*, *r*/*CCCNqNrCrqCCpqCrq* **C*7—*C*3′ *p*/*Cpq*, *q*/*CNqNr*, *r*/*Crq*—8,

8 *CCCNqNpCNqNrCCCNqNrCrqCCpqCrq*.

6′ *p*/*CCNqNpCNqNr*, *q*/*CCNqNrCrq*, *r*/*CCpqCrq* **C*8—*C*5′ *p*/*q*, *q*/*r*—9,

9 *CCCNqNpCNqNrCCpqCrq*.

4′ *q*/*CtCsu*, *r*/*CCtsCtu*, *p*/*CrCsu* **C*8′ *p*/*t*, *q*/*s*, *r*/*u*—10,

10 *CCCrCsuCtCsuCCrCsuCCtsCtu*.

3′ *p*/*CNCsuCtr*, *q*/*CCNCsuNrCNCsuNt*, *r*/*CCrCsuCtCsu* **C*6—*C*9 *q*/*Csu*, *p*/*r*, *r*/*t*—11,

11 *CCNCsuCtrCCrCsuCtCsu*.

3′ *p*/*CNCsuCtr*, *q*/*CCrCsuCtCsu*, *r*/*CCrCsuCCtsCtu* **C*11—*C*10—12,

12 *CCNCsuCtrCCrCsuCCtsCtu*.

3′ *p*/*CNrCsNt*, *q*/*CNCsuCtr*, *r*/*CCrCsuCCtsCtu* **C*4—*C*12—13,

13 *CCNrCsNtCCrCsuCCtsCtu*.

9′ *p*/*CCNrCsNtCCrCsuCCtsCtu*, *q*/*Nv* **C*13—14,

14 *CNvCCNrCsNtCCrCsuCCtsCtu*.

10′ *p*/*v*, *q*/*CCNrCsNtCCrCsuCCtsCtu* **C*14—15,

15 *CCCCNrCsNtCCrCsuCCtsCtuvv*.

4′ $q/CCCNrCsNtCCrCsuCCtsCtuv$, r/v, $p/CpCqp$
$*C15—16$,

16 $CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCCpCqpv$.
6′ $p/CCpCqpCCCNrCsNtCCrCsuCCtsCtuv$, $q/CpCqp$,
r/v $*C16—C9'—17$,

17 $CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvv$.
3′ $p/CCpCqpCCCNrCsNtCCrCsuCCtsCtuv$, q/v, r/Cwv
$*C17—C9'$ p/v, $q/w—18$,

18 $CCCpCqpCCCNrCsNtCCrCsuCCtsCtuvCwv$.

Hence, we have the Lukasiewicz-Tarski thesis.

Therefore, the Lukasiewicz-Tarski axiom 1 is equivalent to the (L_3)-system and the proof is complete.

From above proof line, it is easily seen that classical propositional calculus is completely characterized by the following two theses:

$CpCqp$,

$CCNrCsNtCCrCsuCCtsCtu$.

References

[1] J. Lukasiewicz: Elements of Mathematical Logic (translation from Polish). Oxford (1963).

[2] J. Lukasiewicz und A. Tarski: Untersuchungen über den Aussagenkalkül. C. R. de Varsovie, Cl. III, **23**, 30-50 (1930).

[3] Y. Imai and K. Iséki: On axiom systems of propositional calculi. I. Proc. Japan Acad., **41**, 436-439 (1965).

[4] Y. Arai: On axiom systems of propositional calculi. III. Proc. Japan Acad., **41**, 570-574 (1965).