192. A Duality Theorem for Locally Compact Groups. I

By Nobuhiko TATSUUMA
Department of Mathematics, Kyôto University
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1. Let G be a locally compact group.

The purpose of this paper is to prove the following propositions 1, 2, and 3, which may be considered as an analogue of the Tannaka's duality theorem in the case of compact groups or its homogeneous spaces, in a similar but somewhat narrow sense as [1], [2], and [3].

Let $R=\{R_g\}$, $L=\{L_g\}$ be right and left regular representations respectively, which are realized on the space $L^2(G)$ of all squar summable functions with respect to a right Haar measure μ on G.

If a unitary representation $D=\{U_g, \mathfrak{H}\}$ of G and a complete orthonormal system $\Phi=\{\varphi_\alpha\}_\alpha$ in $\mathfrak{H}$ are given, it is easy to show that the correspondense

$$v\otimes f\to\{\langle U_g v, \varphi_\alpha\rangle f(g)\}_\alpha \tag{1}$$

generates an isometric map A from $\mathfrak{H}\otimes L^2(G)$ onto the discrete direct sum of $L^2(G)$ with multiplicity equal to the dimension of $\mathfrak{H}$, which maps $U_g\otimes R_g$ to direct sum of R_g. I.e.,

Lemma 1. *$D\otimes R$ is unitary equivalent to ΣR_α by the map A, where each R_α is unitary equivalent to R.*

Especially we denote by $A(\Phi)$ the isometric map generated by (1) for the case of $R\otimes R\sim\Sigma R_\alpha$ with respect to the system Φ.

Now we formulate the main propositions.

Proposition 1. *Let T is a unitary operator on $L^2(G)$, satisfying the following conditions:*

$$TL_g=L_gT \quad \text{for any } g \text{ in } G. \tag{2}$$

If $A(\Phi)(f_1\otimes f_2)=\{h_\alpha\}_\alpha$, then

$$A(\Phi)(Tf_1\otimes Tf_2)=\{Th_\alpha\}_\alpha. \tag{3}$$

Then there exists the unique element g_0 in G such that

$$T=R_{g_0}.$$

Let Ω be the set of all equivalence classes of unitary representations of G, and $D=\{U_g^D, \mathfrak{H}^D\}$ be a representative of class $\hat{D}$ in Ω. Consider an operator field $\boldsymbol{T}=\{T(D)\}$ over Ω, where $T(D)$ is a unitary operator in $\mathfrak{H}^D$. Such a $\boldsymbol{T}$ is called *admissible* if the conditions (4) and (5) are satisfied by $\boldsymbol{T}$.

$$U_1(T(D_1)\oplus T(D_2))U_1^{-1}=T(D_3), \tag{4}$$

$$U_2(T(D_1)\otimes T(D_2))U_2^{-1}=T(D_4), \tag{5}$$

for arbitrary map U_1 (resp. U_2) which gives an unitary equivalence between $D_1 \oplus D_2$ (resp. $D_1 \otimes D_2$) and D_3 (resp. D_4).

For any fixed g in G, $\boldsymbol{U}_g = \{U_g(D)\}$ gives an example of such an operator field. And the totality $\mathfrak{R}$ of admissible operator fields becomes a group with the natural multiplication of operators in each component.

By the application to the equivalence classes in Ω which contain R and $R \otimes R$, (4) and (5) lead us to (2) and (3) of proposition 1.

Lemma 2. *Let $\boldsymbol{T}$ be an admissible operator field such that*

$$T(R) = R_g \qquad \text{for some } g \text{ in } G,$$

then

$$\boldsymbol{T} = \boldsymbol{U}_g.$$

In fact, let $\boldsymbol{T}_0 = \boldsymbol{T}\boldsymbol{U}_g^{-1}$, then $\boldsymbol{T}_0$ is an admissible operator field, such that $T_0(R)$ is the identity operator on $L^2(G)$. From lemma 1 and admissibility of $\boldsymbol{T}_0$, we have,

$$(T_0(D)v \otimes f) = (v \otimes f), \qquad \text{for any } v \text{ in } \mathfrak{H}^D \text{ and } f \text{ in } L^2(G).$$

Obviously such an operator $T_0(D)$ must be the identity operator on $\mathfrak{H}^D$. q.e.d.

Proposition 2. *$\mathfrak{R}$ coincides with G; in other words, for any admissible operator field $\boldsymbol{T}$ there is the unique element g_0 in G such that*

$$\boldsymbol{T} = \boldsymbol{U}_{g_0}$$

2. Now we shall prove the proposition 1.

Hereafter we denote by $C_0(G)$ the space of all continuous functions with compact carriers on G.

Lemma 3. *The condition* (3) *results following.*

$$T(hf) = T(h)T(f), \qquad \text{for any } h \text{ in } C_0(G) \text{ and } f \text{ in } L^2(G).$$

In fact, applying (3) to the map (1), in which v, f are any elements of $L^2(G)$, we get for any α,

$$\langle R_g T v, \varphi_\alpha \rangle \, Tf(g) = T(\langle R_g v, \varphi_\alpha \rangle f)(g). \tag{6}$$

Because of arbitrariness of the system Φ, (6) is true even if we replace φ_α by any function φ in $L^2(G)$, so writing the scalar product by integral form and

$$\int (Tv)(g_1 g)\overline{\varphi(g_1)}d\mu(g_1)\, Tf(g) = T\Big(\int v(g_1 g)\overline{\varphi(g_1)}d\mu(g_1) f\Big)(g). \tag{7}$$

Now we take the limit in which φ tends to the Dirac's measure δ on the unit element e of G, for v in $C_0(G)$, then

$$(Tv)(Tf) = T(vf). \tag{8}$$

q.e.d.

Let E be a G_δ-compact set in G. Then there exists ψ_E in $C_0(G)$ such that $0 \leqq \psi_E(g) \leqq 1$, and $E = \{g;\, \psi_E(g) = 1\}$. Then $T(\psi_E^n) \to T(\chi_E)$ $(n \to \infty)$, and for any f in $L^2(G)$, considering ψ_E^n as h in lemma 3, we get

$$T(\chi_E f) = T(\chi_E)T(f), \qquad \text{a.e.}\mu. \tag{9}$$

Especially $T(\chi_E) = T(\chi_E \chi_E) = T(\chi_E)T(\chi_E)$, a.e.$\mu$. I.e., there exists a measurable set $T(E)$, and $T(\chi_E)=\chi_{T(E)}$. And it is immediate that

$$\mu(T(E))=\mu(E), \tag{10}$$

$$T(E_1\cap E_2)\sim[T(E_1)\cap T(E_2)], \tag{11}$$

$$T(gE)\sim gT(E), \tag{12}$$

where $A\sim B$ means that A coincides with B except μ-null set.

Consider the function φ in $C_0(G)$ as follows,

$$\varphi(g)=\int\chi_E(g_1)\chi_E(g_1g)d\mu(g_1)=\mu(E\cap Eg^{-1}). \tag{13}$$

Then from (2),

$$(T\varphi)(g)=\int\chi_E(g_1)(T\chi_E)(g_1g)d\mu(g_1)=\int\chi_E(g_1)\chi_{T(E)}(g_1g)d\mu(g_1)$$
$$=\mu(E\cap T(E)g^{-1}). \tag{14}$$

So $T\varphi$ is a continuous function of g.

Lemma 4. (Iwahori)

$$\|h\|_\infty=\|Th\|_\infty, \quad \textit{for any } h \textit{ in } C_0(G). \tag{15}$$

Denote by $\|\ \|_p$ the norm in $L^p(G)$-space, then

$$\|T(h)\|_{2p}^{2p}=\|T(h)^p\|_2^2=\|T(h^p)\|_2^2=\|h^p\|_2^2=\|h\|_{2p}^{2p},$$

taking the limits of $2p$-roots of both sides for $p\to\infty$ and we get the lemma. q.e.d.

From lemma 4,

$$\sup|T\varphi|=\|T(\varphi)\|_\infty=\|\varphi\|_\infty=\max|\varphi|=\varphi(e)=\mu(E).$$

There are two possibilities.

(1) There exists g_0 such that $T\varphi(g_0)=\mu(E)$.

(2) There exists a diverge sequence $\{g_j\}$ such that $\{T\varphi(g_j)\}$ increase to $\mu(E)$.

But the second case is excluded. In fact, if (2) is true, then for sufficiently large N, $T\varphi(g_j)>(1/2)\mu(E)$, $(j\geqslant N)$, while the compactness of E assures existence of $j(\geqslant N)$ such that $Eg_jg_N^{-1}\cap E=\phi$. So,

$$\mu(E)=\mu(T(E))=\mu(T(E)g_N^{-1})\geqq\mu(E\cap T(E)g_N^{-1})+\mu(Eg_jg_N^{-1}\cap T(E)g_N^{-1})$$
$$=T\varphi(g_N)+\mu(E\cap T(E)g_j^{-1})=T\varphi(g_N)+T\varphi(g_j)$$
$$>(1/2)\mu(E)+(1/2)\mu(E)=\mu(E).$$

That is contradiction.

For the only case (1), $\mu(E\cap T(E)g_0^{-1})=\mu(E)$, and from (10), $E\sim T(E)g_0^{-1}$, therefore we can put

$$T(E)=Eg_0. \tag{16}$$

Let $\{E_\alpha\}$ be a fundamental system of G_δ-compact neighborhoods of e in G. For each E_α, there exists g_α satisfying (16). If $\mu(E_\alpha\cap E_\beta)\neq 0$, then from (11) $0\neq\mu(T(E_\alpha)\cap T(E_\beta))=\mu(E_\alpha g_\alpha\cap E_\beta g_\beta)$, so $g_\alpha g_\beta^{-1}\in E_\alpha^{-1}E_\beta$. This fact means that the family $\mathfrak{F}=\{F_\alpha=\{g_\beta;E_\beta\subseteq E_\alpha\}\}_\alpha$ constructs a base of Cauchy filter in the complete space G. Consequently g_α converges to an element g_T^{-1} in G.

Now consider the function h_α for an arbitrary h in $C_0(G)$,

$$h_\alpha(g)=(1/\mu(E_\alpha))\int h(g_1^{-1})\chi_{E_\alpha}(g_1g)d\mu(g_1),$$

then

$$(Th_\alpha)(g)=(1/\mu(E_\alpha))\int h(g_1^{-1})\chi_{T(E_\alpha)}(g_1g)d\mu(g_1)$$
$$=(1/\mu(E_\alpha))\int h(gg_\alpha^{-1}g_1^{-1})\chi_{E_\alpha}(g_1)d\mu(g_1).$$

Taking the limits of both sides for $E_\alpha\rightarrow\{e\}$, and we get,

$$Th=R_{g_T}h, \qquad \text{for any } h \text{ in } C_0(G).$$

While T is isometric by assumption and C_0 is dense in $L^2(G)$, then

$$Tf=R_{g_T}f, \qquad \text{for any } f \text{ in } L^2(G).$$

This completes the proof of proposition 1.

3. In [2], Mr. and Mrs. Iwahori gave a formulation of an extension of Tannaka duality theorem for homogeneous spaces of compact groups. Here we shall prove similar results for a homogeneous space $H\backslash G$ in which H is a compact subgroup of locally compact group G as in [2].

Denote by N the normalizer of H in G, then any element n in N give a unitary operator on $L^2_\nu(H\backslash G)$, defined by

$$(L_nf)(\tilde{g})=f(\widetilde{ng}), \tag{17}$$

where ν is a quasi-invariant measure on $H\backslash G$, defined by $\nu=\Delta(g)\nu_0$ for $\Delta(g)=(d\mu/d\mu_l)(g)$ (module of right and left Haar measures on G), and invariant measure ν_0 on $H\backslash G$. And $\tilde{g}$ is the image of g by the canonical map π from G to $H\backslash G$. Then a representation $\{U_g\}$ of G on $L^2_\nu(H\backslash G)$ is given by

$$(U_{g_0}f)(g)=(\Delta(g_0))^{1/2}f(gg_0).$$

Proposition 3. *Let $\mathfrak{S}$ be the set of all such unitary operators T on $L^2_\nu(H\backslash G)$ that*

$$T(f_1)T(f_2)=T(f_1f_2), \qquad \text{for any } f_1 \text{ in } C_0(H\backslash G), \text{ and any } f_2 \text{ in } L^2_\nu(H\backslash G), \tag{18}$$

$$TU_g=U_gT, \qquad \text{for any } g \text{ in } G. \tag{19}$$

Then the map B which maps n in N to L_n is a homomorphism of N onto $\mathfrak{S}$, and the kernel of B is H.

Remark. The assumption of proposition 3 follows from the reductivity of T for the Kronecker product between the regular representation of G and the induced representation of G by the trivial representation of H, as in the proof of proposition 1.

Proof of proposition 3. It is sufficient to prove only that B is onto. The same arguments as in 2 are available. And we get the same formulae (10) and (11) for any G_δ-compact set E in $H\backslash G$. Instead of φ in (12), we put,

$$\varphi(\tilde{g})=\mu_l(\widetilde{E}\cap g^{-1}\widetilde{E})=\int\chi_{\widetilde{E}}(g_0)\chi_{\widetilde{E}}(gg_0)d\mu_l(g_0),$$
$$\text{for the set } \widetilde{E}=\pi^{-1}(E), \tag{20}$$

and consider $T\varphi$ in the same way. So we can conclude the existence of g_E in G such that

$$\widetilde{T(\widetilde{E})} \sim g_E \widetilde{E}, \tag{21}$$

but $\widetilde{T(\widetilde{E})}$ is a H-cosetwise set in G then for any h in H,

$$h g_E \widetilde{E} \sim h \widetilde{T(\widetilde{E})} \sim \widetilde{T(\widetilde{E})} \sim g_E \widetilde{E},$$

this means

$$g_E^{-1} H g_E \subset \widetilde{E} \widetilde{E}^{-1}.$$

Let $\{\hat{E}_\alpha\}$ be a fundamental system of G_δ-compact neighborhoods of e in G. There is an element g_α corresponding to $E=\pi(\hat{E}_\alpha)$ which satisfies (21).

Lemma 5. *For any given neighborhood W of e in G, there is a set $\hat{E}_\alpha$ such that*

$$H \hat{E}_\alpha \hat{E}_\alpha^{-1} H \subset HW. \tag{22}$$

In fact, for any h in H there is a $\hat{E}_h$ such that

$$h \hat{E}_h \hat{E}_h \hat{E}_h^{-1} h^{-1} \subset HW.$$

From compactness of H, there is a finite covering $\{h_j \hat{E}_{h_j}\}$ of H, put

$$\hat{E}_\alpha \subset \bigcap_j \hat{E}_{h_j},$$

then we get the result immediately. q.e.d.

For $\hat{E}_\alpha$ as (22) and $\hat{E}_\beta$ contained in $\hat{E}_\alpha$,

$$g_\beta H \hat{E}_\beta \cap g_\alpha H \hat{E}_\alpha \neq \phi,$$

so,

$$g_\alpha^{-1} \in H \hat{E}_\alpha \hat{E}_\beta^{-1} H g_\beta^{-1} \subset H \hat{E}_\alpha \hat{E}_\alpha^{-1} H g_\beta^{-1} \subset HW g_\beta^{-1},$$

i.e., the set $\{F_\alpha = \{\pi(g_\beta^{-1});\ \hat{E}_\beta \subset \hat{E}_\alpha\}\}_\alpha$ constructs a base of Cauchy filter in $H \backslash G$. The completeness of $H \backslash G$ assures the existence of unique limit $\widetilde{g}_T$ of this filter. Moreover from lemma 5, for any W there exists $\hat{E}_\alpha$ such that

$$g_\alpha^{-1} H g_\alpha \subset H \hat{E}_\alpha \hat{E}_\alpha^{-1} H \subset HW,$$

therefore taking the limit, we get

$$g_T H g_T^{-1} \subset H, \qquad \text{i.e. } g_T = n \in N.$$

The same argument as in 2 for the function

$$h_\alpha(\widetilde{g}) = (1/\mu(H\hat{E}_\alpha)) \int h(\pi(g_0^{-1})) \chi_{H\hat{E}_\alpha}(g g_0) d\mu_l(g_0),$$

leads us to

$$Th = L_n h.$$

This formula can be extended onto $L_\nu^2(H \backslash G)$ easily.

So proposition 3 is proved.

References

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