2. Remarks on Periodic Solutions of Linear Parabolic Differential Equations of the Second Order

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1. **Introduction.** Let E^m be the m-dimensional Euclidian space of points $x=(x_1, \cdots, x_m)$ and let Ω be an unbounded domain in E^m with boundary $\partial\Omega$. We set $Q=\{(x,t): x\in\Omega, -\infty<t<\infty\}$ and $\partial Q=\{(x,t): x\in\partial\Omega, -\infty<t<\infty\}$. Q is an infinite cylinder in E^{m+1} whose base is Ω and whose (lateral) boundary is ∂Q. $\bar{Q}$ denotes the closure of Q.

In this note we shall be concerned with periodic solutions of the first boundary problem in Q for linear second order parabolic equations having periodic coefficients and right members.[1]

We shall briefly discuss the existence and the uniqueness of the periodic solutions which may grow exponentially as the variable x tends to infinity.

In our discussion we shall use the method similar to that employed by M. Krzyżański in regard to elliptic and parabolic boundary problems in unbounded domains [1–3].

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2. Let us consider the equation.

$$(1)\qquad Lu=\sum_{i,j=1}^{m} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{m} b_i(x,t)\frac{\partial u}{\partial x_i}+c(x,t)u-\frac{\partial u}{\partial t} = f(x,t) \quad \text{in } Q,$$

and the boundary condition

$$(2)\qquad u(x,t)=\varphi(x,t) \quad \text{on } \partial Q.$$

We shall need the following assumptions:

1°. The functions a_{ij}, b_i, c, f, and φ are continuous in $\bar{Q}$ and periodic with period T $(T>0)$.

2°. There exist positive constants A, B, and C such that

$$|a_{ij}|\leqq A,\ |b_i|\leqq B,\ c\leqq -C \quad \text{in } \bar{Q}.$$

3°. The form $\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j$ is positive definite in $\bar{Q}$.

Definition. We shall say that a function $w(x,t)$ belongs to class $\bar{E}_1(K)$ $(\underline{E}_1(K))$ if there exist positive constants M_0 and k_0 $(0<k_0<K)$ such that

1) Here and throughout by a *periodic* function is meant one which is periodic in the time variable t.

$$w(x,t)\leqq M_0\exp\Big(k_0\sum_{i=1}^{m}|x_i|\Big)\quad\Big(w(x,t)\geqq -M_0\exp\Big(k_0\sum_{i=1}^{m}|x_i|\Big)\Big)$$

in $\bar{Q}$. We denote by $E_1(K)$ the class of functions belonging to $\bar{E}_1(K)$ and $\underline{E}_1(K)$ simultaneously.

Our first result is the following maximum principle.

Theorem 1. *Let $u(x,t)$ be a regular*[2] *periodic (period T) solution of the problem* (1), (2) *belonging to class $\bar{E}_1(K)$ ($\underline{E}_1(K)$), where K is the positive root of the equation (in k)*

(3) $$m^2Ak^2+mBk-C=0.$$

If $f(x,t)\geqq 0$ ($\leqq 0$) in $\bar{Q}$ and $\varphi(x,t)\leqq 0$ ($\geqq 0$) on ∂Q, then $u(x,t)\leqq 0$ ($\geqq 0$) in $\bar{Q}$.

Proof. We introduce the function

(4) $H(x:k)=\prod_{i=1}^{m}\cosh kx_i$ (k: a positive parameter)

constructed by M. Krzyżański [3]. It has the following properties: (i) $2^{-m}\exp\Big(k\sum_{i=1}^{m}|x_i|\Big)<H(x:k)<\exp\Big(k\sum_{i=1}^{m}|x_i|\Big)$: (ii) if $0<k<k'$, then $H(x:k)/H(x:k')\to 0$ as $x\to\infty$: (iii) to each k, $0<k<K$, there corresponds a number $\delta(k)>0$ such that $LH(x:k)\leqq -\delta(k)H(x:k)$ in Q.

We denote by Q_N the intersection of Q with the circular cylinder $\{(x,t):|x|<N, -\infty<t<\infty\}$. The boundary ∂Q_N consists of two parts: the part $S_N^{(1)}=\partial Q\cap\partial Q_N$ and the remaining part $S_N^{(2)}$.

Now, by hypothesis, there are constants M_0 and k_0 ($0<k_0<K$) such that $u(x,t)\leqq M_0\exp(k_0\sum_{i=1}^{m}|x_i|)$ in $\bar{Q}$.

Consider the function $v(x,t)$ defined by $v(x,t)=u(x,t)/H(x:\bar{k})$ ($0<k_0<\bar{k}<K$). Given an arbitrary number $\varepsilon>0$, we can choose $N>0$ so large that

(5) $v(x,t)<\varepsilon$ on ∂Q_N.

This follows from the following fact: $v(x,t)$ is non-positive on $S_N^{(1)}$ by the prescribed boundary condition, whereas it satisfies on $S_N^{(2)}$ the following inequalities:

$$v(x,t)\leqq M_0\exp\Big(k_0\sum_{i=1}^{m}|x_i|\Big)/H(x:\bar{k})$$
$$=2^m M_0\exp\Big(-(\bar{k}-k_0)\sum_{i=1}^{m}|x_i|\Big).$$

Observing that $v(x,t)$ satisfies in Q_N the linear equation

$$\bar{L}v=\sum_{i,j=1}^{m}a_{ij}\frac{\partial^2 v}{\partial x_i\partial x_j}+\sum_{i=1}^{m}\bar{b}_i\frac{\partial v}{\partial x_i}+\bar{c}v-\frac{\partial v}{\partial t}=\bar{f},$$

where

$$\bar{c}=LH(x:\bar{k})/H(x:\bar{k})\leqq -\delta(\bar{k})<0\quad\text{and}\quad \bar{f}=f/H(x:\bar{k})\geqq 0.$$

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2) A function $w(x,t)$ is called regular if it is continuous in $\bar{Q}$ and if it possesses the derivative $\partial w/\partial t$ and the continuous derivatives $\partial w/\partial x_i$, $\partial^2 w/\partial x_i\partial x_j$ in Q.

By the usual maximum principle of parabolic type equation, we conclude that the inequality (5) holds true throughout $\bar{Q}_N$. Let (x', t') be an arbitrary point of Q. It lies in Q_N for sufficiently large N, so that $v(x', t') < \varepsilon$. In view of the arbitrariness of ε we have $v(x', t') \leqq 0$, whence we assert that $v(x, t) \leqq 0$ and hence $u(x, t) \leqq 0$ in $\bar{Q}$.

Corollary. *The boundary problem* (1), (2) *has at most one regular periodic* (*period* T) *solution belonging to class* $E_1(K)$.

In the particular case where $\Omega = E^m$, that is, Q coincides with the entire space E^{m+1}, we get the following Theorem 1′.

Theorem 1′. *If* $u(x, t)$ *is a regular periodic* (*period* T) *function which is of class* $\overline{E}_1(K)$ ($\underline{E}_1(K)$) *in* E^{m+1} *and such that* $Lu \geqq 0$ ($\leqq 0$) *in* E^{m+1}, *then* $u(x, t) \leqq 0$ ($\geqq 0$) *throughout the space* E^{m+1}.

3. This paragraph is devoted to the study of the existence of periodic solutions of the boundary problem (1), (2).

Hypothesis (H). Let $\psi(x, t)$ be an arbitrary continuous function in $\bar{Q}$ which is periodic (period T). For every $N > 0$ there exists a regular periodic (period T) solution $u(x, t)$ of the equation (1) in Q_N satisfying the boundary condition $u(x, t) = \psi(x, t)$ on ∂Q_N.

Theorem 2. *Let the hypothesis* (H) *be satisfied. If, in addition to the assumptions* 1°—3° *already made, we assume the following:*

4°. *The functions* f *and* Φ *are continuous in* $\bar{Q}$, *periodic* (*period* T) *and belong to class* $E_1(K)$. $\Phi(x, t)$ *is the extension of* $\varphi(x, t)$ *and* $\Phi(x, t) = \varphi(x, t)$ *on* ∂Q. *Then the problem* (1), (2) *has a unique regular solution which is periodic* (*period* T) *and belongs to class* $E_1(K)$.

Proof. At first, we construct, according to the hypothesis (H), a sequence of periodic (period T) functions $u_N(x, t)$ satisfying:

$Lu_N = f(x, t)$ in Q_N and $u_N(x, t) = \Phi(x, t)$ on ∂Q_N ($N = 1, 2, \cdots$).

To show the convergence of this sequence we introduce the functions

$$v_N(x, t) = u_N(x, t)/H(x : k^*) \qquad (N = 1, 2, \cdots).$$

$v_N(x, t)$ satisfies the linear equation of the form

$$(6) \qquad L^* v_N = \sum_{i,j=1}^{m} a_{ij} \frac{\partial^2 v_N}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i^* \frac{\partial v_N}{\partial x_i} + c^* v_N - \frac{\partial v_N}{\partial t} = f^* \text{ in } Q_N$$

and the boundary condition

$$(7) \qquad v_N(x, t) = \Phi^*(x, t) = \Phi(x, t)/H(x : k^*) \text{ on } \partial Q_N.$$

From the assumption 4° there are positive numbers M_0 and k_0 $(0 < k_0 < K)$ such that

$$|f(x, t)| \leqq M_0 \exp\Big(k_0 \sum_{i=1}^{m} |x_i|\Big), \ |\Phi(x, t)| \leqq M_0 \exp\Big(k_0 \sum_{i=1}^{m} |x_i|\Big) \text{ in } \bar{Q}.$$

If k^* is such that $0 < k_0 < k^* < K$, then we find that

$$c^*(x, t) \leqq -\delta(k^*) < 0, \ |f^*(x, t)| \leqq 2^m M_0 \text{ and } |\Phi^*(x, t)| \leqq 2^m M_0 \text{ in } \bar{Q}.$$

We put $w_N^{\pm}(x, t)=2^m M_0(1+1/\delta(k^*))\pm v_N(x, t) \qquad (N=1, 2, \cdots)$.

Since $w_N^{\pm}$ satisfy the inequalities $L^* w_N^{\pm}\leqq 0$ in Q_N and $w_N^{\pm}(x,t)\geqq 0$ on ∂Q_N, by the maximum principle we get $w_N^{\pm}\geqq 0$ in $\bar{Q}_N$, or equivalently,

(8) $\quad |v_N(x, t)|\leqq 2^m M_0(1+1/\delta(k^*))=M_1$ in $\bar{Q}_N$ $(N=1, 2, \cdots)$.

We set

(9) $\quad w_{NN'}=(u_N-u_{N'})/H(x:k^{**})=(v_N-v_{N'})H(x:k^*)/H(x:k^{**})$,

where $N<N'$ and $0<k_0<k^*<k^{**}<K$. $w_{NN'}(x, t)$ satisfies in Q_N a homogeneous equation analogous to (6). For any given $\sigma>0$ there is an N such that $|w_{NN'}(x, t)|<\sigma$ on ∂Q_N (This follows readily from (8), (9) and the property (ii) of $H(x:k)$). Hence, we get $|w_{NN'}(x,t)|<\sigma$ in $\bar{Q}_N$, whence

$$|u_N(x, t)-u_{N'}(x, t)|<\sigma \operatorname*{l.u.b.}_{\bar{Q}'} H(x:k^{**}) \text{ in } \bar{Q}_N$$

Q' being an arbitrary cylinder contained in $\bar{Q}_N$. This shows that the sequence $\{u_N(x, t)\}$ is uniformly convergent in every cylinder with bounded base in $\bar{Q}$. Clearly, the limit function $u(x, t)=\lim_{N\to\infty} u_N(x, t)$ is periodic (period T) and takes on the boundary values $\varphi(x, t)$ on ∂Q.

It remains to show that $u(x, t)$ is a regular solution of (1). It is enough to prove this in the cylinder Q_{N_0} for an arbitrary N_0.

To this end, let $U(x, t)$ be a regular periodic (period T) solution of (1) in Q_{N_0} such that $U(x, t)=u(x, t)$ on ∂Q_{N_0}. Given any $\varepsilon>0$, there is an $N_1>N_0$ such that for $N>N_1$ we have

(10) $\qquad |U(x, t)-u_N(x, t)|<\varepsilon$ on ∂Q_{N_0}

(11) $\qquad |u(x, t)-u_N(x, t)|<\varepsilon$ in Q_{N_0}.

We set $V(x, t)=U(x, t)/H(x:k^*)$ and denote by $\Gamma(k^*)$ and $\gamma(k^*)$ $\operatorname*{l.u.b.}_{\bar{Q}_{N_0}} H(x:k^*)$ and $\operatorname*{g.l.b.}_{\bar{Q}_{N_0}} H(x:k^*)$, respectively. Noting that $V(x, t)-v_N(x, t)$ satisfies a homogeneous equation analogous to (6) and that it is less than $\dfrac{\varepsilon}{\gamma(k^*)}$ on ∂Q_{N_0} (see (10)), we obtain

$$|V(x, t)-v_N(x, t)|<\frac{\varepsilon}{\gamma(k^*)} \text{ in } Q_{N_0}.$$

This implies that

(12) $$|U(x, t)-u_N(x, t)|<\varepsilon\frac{\Gamma(k^*)}{\gamma(k^*)} \text{ in } Q_{N_0}.$$

From (11) and (12) we obtain

$$|U(x, t)-u(x, t)|<\varepsilon\left\{\frac{\Gamma(k^*)}{\gamma(k^*)}+1\right\} \text{ in } Q_{N_0}$$

which means that $U(x, t)=u(x, t)$ in $\bar{Q}_{N_0}$.

That the solution $u(x, t)$ belongs to class $E_1(K)$ is an immediate consequence of (8):

$$|u(x, t)|\leqq M_1 H(x:k^*)\leqq M_1 \exp\left(k^*\sum_{i=1}^{m}|x_i|\right) \text{ in } \bar{Q}.$$

The uniqueness of the periodic solution follows from Theorem 1.

The proof is thus completed.

Remark. From Theorem 1 of I. I. Shmulev [4] it is not difficult to point out a situation where the *hypothesis* (H) is actually satisfied under some conditions.

We conclude by stating the following theorem on the existence of entire periodic solutions.

Theorem 2′. *Let the following assumptions be satisfied:*

I. *The coefficient* $a_{ij}, b_i,$ *and* c *of* (1) *are periodic with period* T *and locally Hölder continuous in* E^{m+1}. *There are positive constants* $A, B,$ *and* C *such that*

$$|a_{ij}|\leqq A,\quad |b_i|\leqq B,\quad c\leqq -C \quad \textit{in} \quad E^{m+1}.$$

II. *There exists a positive constant* μ *such that*

$$\sum_{i,j=1}^{m} a_{ij}\xi_i\xi_j \geqq \mu \sum_{i=1}^{m} \xi_i^2 \quad \textit{in} \ E^{m+1}.$$

III. *The function* f *is locally Hölder continuous in* E^{m+1}, *periodic with period* T *and belongs to class* $E_1(K)$, *where* K *is the positive root of* (3).

Then there exists one and only one periodic (*period* T) *function* $u(x, t)$ *satisfying the equation* (1) *in* E^{m+1} *and belonging to class* $E_1(K)$.

References

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