30. E.R. ν Singular Cut-Off by the Measure $\nu(x, \delta; \Lambda)$

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§1. **Introduction:** Singular cut-off by using the measure defined in Minkovski space has shown in [3] p. 550. The advantage of this E.R. ν singular cut-off is to admit the expectation value conserving transform of non-local field function (related to Lorentz transform) satisfying the formal Lorentz covariance. In this transform all non-covariant effects are reduced to the change of the definition of singular integral (the change of the measure ν). Since this change of ν corresponds to the change of summation's order, it can be considered as the change (related to non-local structure) by the concepts independent of Lorentz transform. As a possible deformation of this singular cut-off, in §2 we give a sort of three dimensional singular cut-off by using the measure $\nu(x, \delta; \Lambda)$ (a deformed singular cut-off related to the neighbourhood of the set of all space-like position for a point). Since this new singular cut-off has the simple form near the three dimensional one, it seems that this replacement of ν depending on Λ can be understood by non-local mechanism well.

The non-local field function by this singular cut-off satisfies the expectation value conserving Lorentz covariance by the form

$$U(a, \Lambda)(1/2\sigma)\left\{\text{E.R. } B.\, \nu(x, \delta; 1)\iint_{\{x';x'^2=x_0'^2-\vec{x}'^2\leqslant\sigma^2\}} f(x')\, dx_0'\, \varphi(x-\vec{x}')\, d\vec{x}'\right\} U^{-1}(a, \Lambda)$$

$$=(1/2\sigma)\left\{\text{E.R. } B.\, \nu(x, \delta; \Lambda)\iint_{\{x';x'^2=x_0'^2-\vec{x}'^2\leqslant\sigma^2\}} f(x')\, d(\Lambda(x_0'))\, \varphi(\Lambda x+a-\Lambda(\vec{x}'))\, d(\Lambda(\vec{x}'))\right\} \quad (1)$$

(see §3 Def. 4). $\int d(\Lambda(x_0'))$ is the normalized one satisfying

$$\iint d(\Lambda(x_0')\, d(\Lambda(\vec{x}'))=\int d(\Lambda x').$$

Here, the integral related to this cut-off is by the meaning of [2] p. 377 Def. 1, and E.R. $B\ \nu\int$ is a special form of E.R. $\nu\int$ defined in [2] p. 548 Def. 2. This covariance has the advantage similar to the results in [9] p. 35 which is the origin of the Λ inhomogeneous Lorentz covariance (for three dimensional case) appearing in [2] p. 380 Def. 3. Because we can obtain the various initial conditions related to (a, Λ) (on the various space-like manifolds) from this Lorentz covariant (unified) form (1). Furthermore this measure

$\nu(x, \delta; \Lambda)$ is based on an interpretation (an hypothesis) of E.R. ν singular cut-off (by probability). It seems that this interpretation replaces the freedom related to r_μ in [10] p. 220 by the suitable statistical interpretation of the behavior of positive and negative components in "elementary particle". Furthermore it assures the rightousness of the model of moving $\delta(x)s$ near the one in [1] p. 74.

$\nu(x, \delta; \Lambda)$ is the measure with the following properties;

(i) $\nu(x, \delta; \Lambda)=\nu_\Lambda(x, \delta; 1)$

$$\left([3] \text{ p. } 551 \quad \nu_\Lambda(x, \delta; 1)(B)=\int_B \Phi(\Lambda^{-1}x; \delta, 1)\,dx\right),$$

(ii) the mean (see § 2 Def. 4) of $f(x)$ by this ν in the set $\{x; x^2=x_0^2-\vec{x}^2\leqslant l^2\}$ constructs the mollifier corresponding to $g(\vec{x})\in(z)$ $((\mathfrak{S})$ or $(\mathfrak{D}))$ for any observer, and this mean in $\{x; x^2=x_0^2-\vec{x}^2\leqslant\sigma^2\}$ $(0\leqslant\sigma<l)$ constructs the mollifier corresponding to $\delta(\vec{x})$, where the singular function $f(x)$ has the property $f(x)\equiv f(\Lambda x)$ in [3] p. 549 Lemma 4.

This $\nu(x, \delta; \Lambda)$ seems to be effective to give the more suitable connection connected with relativity between the local field theory and non local field theory. Then in § 3 by using a sort of uncertainty about the relativistic distance not contradict to well known one by Heisenberg, let's discuss more precisely (than [3] p. 551) the interpretation of commutation relation of these non-local fields in connection with macrocausality by using the property (ii) of $\nu(x, \delta; \Lambda)$.

§ 2. Interpretation of E.R. ν singular cut-off by probability: The definition of E.R. ν integral [2] p. 547 Def. 1 and its special form E. R. B. ν integral corresponding to one defined in [2] p. 548 Def. 2 are not shown but used here. According to the proof by H. Okano, E. R. B. $\nu\int$ is the special one of E.R. $\nu\int$.

Singular cut-off is to construct a sort of singular integral convolution (or its deformation) of field function by a singular mollifier [2] p. 377 Def. 1, [3] p. 548. This convolution is deduced from the similar thought to [8] p. 391 etc.

Definition 1. If E.R. integral [6] p. 17 (equal to A integral [5] p. 131 which has the original form (B) integral [4] p. 220) is used as singular integral in this cut-off, this singular cut-off is called E.R. singular cut-off (equal to A singular cut-off). Similar definitions are given to cut-offs in which various singular integrals are used.

The notations used in this paragraph are the following. X; a Minkovski space E^4 with a fixed coordinate. ΛX; Minkovski space E^4 with the coordinate transformed by Λ (homogeneous Lorentz transform) from one defined in X. S_k; the set in E^4

$$\{x; (\sum_{p=0,p\geqslant 0}^{k-3} 3^{-p})^2 < x^2=x_0^2-\vec{x}^2\leqslant(1+\sum_{p=1,p\geqslant 1}^{k-2} 3^{-p})^2\}$$

for $k \geqslant 2$, and $\{x; x^2 = x_0^2 - \vec{x}^2 \leqslant 0\}$ for $k=1$. $f(x)$; singular function defined in E^4 with the property defined in [3] p. 549 Lemma 4 and with $f(x) \equiv f(\Lambda x)$ for any Λ. $C(n, +, \Lambda)$; nth class of positive particles (components) by the observer in ΛX. $C(n, -, \Lambda)$; nth class of negative particles (components) by the observer in ΛX. $C(n, \Lambda) = C(n, +, \Lambda) \cup C(n, -, \Lambda)$.

Hereafter we will show only the notation related to positive particles for the simplicity. But we use the similar notations related to negative particles without any descriptions. $\nu(\delta, k, \Lambda)$ $(k=1, 2, \cdots)$; the measures (depending on $f(x)$) which satisfy the relation $(3^{k-1}/2)$ E. R. B. $\nu(\delta, k, \Lambda)$

$$\iint_{S_k} f(x)\, d(\Lambda(x_0))\, \tilde{\psi}_\Lambda(\Lambda(\vec{x}))\, d(\Lambda(\vec{x})) = \int \delta(\vec{x})\, \psi(\vec{x})\, d\vec{x}$$

for any Λ, any real a, and any $\psi(\vec{x}) \in (\mathfrak{B})$ [7], II, p. 55, where $\tilde{\psi}_\Lambda(x)$ is the function defined in the set $\{x; x = \Lambda \vec{x}, \vec{x} \in E^3\}$ (for fixed Λ) with the property $\tilde{\psi}_\Lambda(\Lambda(\vec{x})) \equiv \psi(\vec{x})$. $\nu(\delta, k, \Lambda)$ can be constructed by the same method as the construction of ν_Λ in [3] p. 551. $X_{n,k}^+(\Lambda)$; the set of x in which the positive particles contained in $C(n, +, \Lambda)$ appears under the rule by $\nu(\delta, k, \Lambda)$. $SX_{n,k}^+(\Lambda) \equiv \cup_{m \geqslant n} X_{m,k}^+(\Lambda)$. $[f]_{n,k,\Lambda}(x)$; the function in nth step (defined in [3] p. 547 Def. 1) by $\nu(\delta, k, \Lambda)$, $[f]_{0,k,\Lambda}(x) \equiv 0$,

$$f(x; n, k, \Lambda, +) \equiv [[f]_{n,k,\Lambda}(x)]^+ - [[f]_{n-1,k,\Lambda}(x)]^+$$

etc., where $[\ \ ]^+$ means the positive part of function.

Now let's show the hypothesis by which the fundamental meaning of a sort of three dimensional E.R. ν singular cut-off discussed in introduction is given. This hypothesis is derived from the expectation of positive and negative particles (the component of "elementary particle") under the complicated Brownian motion with many stable points, and it (the interpretation by probability) is shown here by the construction of the following special model. (We can generalize this model easily.)

(A) (the properties of $X_{n,k}^+(\Lambda)$ and $X_{n,k}^-(\Lambda)$)

(i) $SX_{k,k}^+(\Lambda) \cup SX_{k,k}^-(\Lambda) = S_k$ and $X_{n,k}^+(\Lambda) = X_{n,k}^-(\Lambda) = \phi$ for $k > n$.

(ii) $SX_{k,k}^+(\Lambda) \supset SX_{k+1,k}^+(\Lambda) \supset \cdots$,
$SX_{k,k}^-(\Lambda) \supset SX_{k+1,k}^-(\Lambda) \supset \cdots$.

(iii) $(SX_{k,k}^+(\Lambda) \cup SX_{k,k}^-(\Lambda)) \cap (SX_{k',k'}^+(\Lambda) \cup SX_{k',k'}^-(\Lambda)) = \phi$ for any $k \neq k'$, $SX_{k,k}^+(\Lambda) \cap SX_{k,k}^-(\Lambda) = \phi$ for any k, and $\cup_{\pm} \cup_k SX_{k,k}^{\pm}(\Lambda) = F \cap \{x; x^2 = x_0^2 - \vec{x}^2 \leqslant (\sum_{p=0}^{\infty} 3^{-p})^2 = l^2\}$, where $F = \{x; f(x) \neq 0\}$.

(B) (the properties of the summation by n related to $\delta(\vec{x})$)

(i) If $k \geqq 2$, $SX_{n,k}^{\pm}(1)$ contains the set $SX_{k,k}^{\pm}(1) \cap \{x; (1 + \sum_{p=1,\, p\geqslant 1}^{k-2} 3^{-p} - \varepsilon_{n,k}(\|\vec{x}\|^2))^2 \leqslant x^2 = x_0^2 - \vec{x}^2 < (1 + \sum_{p=1,\, p\geqslant 1}^{k-2} 3^{-p})^2\}$ for suitable non increasing sequence (by n) of positive functions $\varepsilon_{n,k}(\|\vec{x}\|^2)$, and if $k=1$, it contains the set $SX_{1,1}^{\pm}(1) \cap \{x; x^2 = x_0^2 - \vec{x}^2 > -\varepsilon_{n,1}(\|\vec{x}\|^2)\}$ for suitable non increasing sequence of non negative functions $\varepsilon_{n,1}(\|\vec{x}\|^2)$.

(ii) $f(x; n, k, \Lambda, +)\neq 0$ in only bounded $X^+_{n,k}(\Lambda)$ and $f(x; n, k, \Lambda, -)\neq 0$ in only bounded $X^-_{n,k}(\Lambda)$ (for $n\geqslant k$).

(iii) For a fixed n, probability appearing in each point x and total density related to positive (negative) particles are defined by

$$\sum_{k=1}^{n} f(x; n, k, \Lambda, +)/\sum_{k=1}^{n}\int_{X^+_{n,k}(\Lambda)} f(x; n, k, \Lambda, +)dx$$

and
$$\sum_{k=1}^{n}\int_{X^+_{n,k}(\Lambda)} f(x; n, k, \Lambda, +)dx$$

$$\left(\sum_{k=1}^{n} f(x; n, k, \Lambda, -)/\sum_{k=1}^{n}\int_{X^-_{n,k}(\Lambda)} f(x; n, k, \Lambda, -)dx\right.$$

and
$$\left.\sum_{k=1}^{n}\int_{X^-_{n,k}(\Lambda)} f(x; n, k, \Lambda, -)dx\right).$$

From the definition of $f(x; n, k, \Lambda, +)$, $\nu(\delta, k, \Lambda)$ etc. this sum by n constructs δ-like function.

(C) (the summation by (n, k) related to $g(\vec{x})$)

$$(1/2l)\sum_{n=1}^{\infty}\left\{\sum_{k=1}^{n}\iint f(x; n, k, \Lambda, +)\, d(\Lambda(x_0))\, \tilde{\psi}_\Lambda(\Lambda(\vec{x}))\, d(\Lambda(\vec{x}))\right.$$
$$\left.+\sum_{k=1}^{n}\iint f(x; n, k, \Lambda, -)\, d(\Lambda(x_0))\, \tilde{\psi}_\Lambda(\Lambda(\vec{x}))\, d(\Lambda(\vec{x}))\right\}=\int g(\vec{x})\,\psi(\vec{x})\,d\vec{x}$$

for any $g(\vec{x})\in(Z)$ (($\mathfrak{S}$) or ($\mathfrak{D}$)) and any element $\psi(\vec{x})$ ($\tilde{\psi}_\Lambda(\Lambda(\vec{x}))$) in ($\mathfrak{B}$).

These requirements (A)-(C) may be satisfied by probability meaning for each (and for one) so-called elementary particle belonging to the same kind.

Definition 2. The concrete model satisfying the above requirements (A)-(C) is called fundamental model.

In the order of $k=1, 2, 3, \cdots$, we can construct $\{f(x; k, k, 1, +)\}$ $\{f(x; k, k, 1, -)\}$ and $\nu(\delta, k, 1)$ satisfying the above requirements (A)-(C) for $\Lambda=1$ from $f(x)$ [3] p. 550 Theorem 1.

Since $\partial(\Lambda x)/\partial(x)=1$, we assert the following theorem from the similar method to the construction of ν_Λ[3] p. 551.

Theorem 1. *$\nu(\delta, k, \Lambda)$ $(k=1, 2, 3, \cdots)$ $X^+_{n,k}(\Lambda)$ and $X^-_{n,k}(\Lambda)$ satisfying the requirements* (A)-(C) *can be constructed.*

In the proof of this theorem, the change of $d(\Lambda(x_0))$ and $d(\Lambda(\vec{x}))$ by Λ plays the very essential role.

By a sort of periodicity the non-local range $\{x; x^2=x_0^2-\vec{x}^2\leqslant l^2\}$ can extend till E^4. We can also construct new model in which l (corresponding to $\sum_{p=0}^{\infty}3^{-p}$) becomes ∞. These models easily become non relativistic form by the consideration of infinite light velocity.

Definition 3. If $X^+_{n,k}(\Lambda)$ and $X^-_{n,k}(\Lambda)$ are the sets contained in $F\cap S_k$ which plays the same role as (B) (i), the measures with the properties (A)-(C) is denoted by $\nu(x, \delta; \Lambda)$.

Theorem 2. *$\nu(x, \delta; \Lambda)$ satisfies the conditions* (i) (ii) *shown in introduction.*

Definition 4. The singular cut-off by $\nu(x, \delta; \Lambda)$ is to construct the following non-local field function $\varphi * f$ from the local field function $\varphi(x)$ by using $f(x)$;

$$\varphi * f = (1/(2\pi)^{3/2})$$

$$\left\{ \int (a^{+}(\vec{k})/\sqrt{2k_0}) \cdot \mathrm{E.R.}B.\ \nu(x, \delta; \Lambda) \iint_{\{x; x^2 = x_0^2 - \vec{x}^2 \leqslant \sigma^2\}} f(x')\, d(\Lambda(x_0'))\right.$$

$$\exp i\{\vec{k}\Lambda(\vec{x}-\vec{x}') + ka - k_0 \Lambda t\}\, d(\Lambda(\vec{x}'))\, d\vec{k} + \int (a(\vec{k})/\sqrt{2k_0}) \cdot \mathrm{E.R.}B. \nu(x, \delta; \Lambda)$$

$$\left. \iint_{\{x; x^2 = x_0^2 - \vec{x}^2 \leqslant \sigma^2\}} f(x')\, d(\Lambda(x_0')) \exp(-i)\{\vec{k}\Lambda(\vec{x}-\vec{x}') + ka - k_0 \Lambda t\}\, d(\Lambda(\vec{x}'))\, d\vec{k} \right\}$$

for (a, Λ).

Here $\int d(\Lambda(x_0))$ means the integral along the pararell axes to $\tau_1 \equiv \{x; x = \Lambda(\vec{0}, x_0)\}$ under the normalization defined in introduction whose value takes at the meet point of these axes and the hyperplane $\Sigma_\Lambda \equiv \{x; x = \Lambda(\vec{x}, 0)\}$, and $\int d(\Lambda(\vec{x}'))$ means the integral in Σ_Λ.

According to (B)-(C), for the model by $\nu(x, \delta; \Lambda)$ (for any Λ), it seems that local theory is satisfied for the mollifier defined by the mean (Def. 4) of $f(x)$ in $\{x; x^2 \leqslant 0\}$ (or $\{x; x^2 = x_0^2 - \vec{x}^2 \leqslant \sigma^2\}$ $(0 < \sigma < l)$) and non-local theory is satisfied for the mollifier defined by the mean (Def. 4) in $\{x; x^2 = x_0^2 - \vec{x}^2 \leqslant l^2\}$. Then Lorentz covariance and causality condition seems to be able to satisfy by the nearly ordinary meaning.

§3. The role of the measure $\nu(x, \delta; \Lambda)$ in commutation relation; By using this $\nu(x, \delta; \Lambda)$, let's show here that the macro-causality is satisfied by this non-local theory in a sense. Here at the first step we use the most usual causality condition as follows [8] p. 390;

Definition 5. If $[\varphi(x), \Pi(x')] = 0$ holds valid for space-like pair of points (x, x'), then $\varphi(x)$ satisfies the causality condition.

At the first step, let's consider the generalized operator valued function $(1/2\sigma)^2$ E. R. $B.$ $\nu(x^*, \delta; \Lambda) \cdot$ E. R. $B.$ $\nu'(x^{*\prime}, \delta; \Lambda')$

$$\iiiint_{\{(x^*, x^{*\prime}); x^{*2} = x_0^{*2} - \vec{x}^{*2} < \sigma^2, (x^{*\prime} - x' + x)^2 = (\ )_0^2 - (\vec{\ })^2 < \sigma^2\}} h(x^*, x^{*\prime})\, d(\Lambda(x_0^*))\, d(\Lambda(x_0^{*\prime}))$$

$$[\varphi(\Lambda x + a - \Lambda(\vec{x}^*)), \Pi(\Lambda x' + a - \Lambda(\vec{x}^{*\prime}))]\, d(\Lambda(\vec{x}^*))\, d(\Lambda(\vec{x}^{*\prime})).$$

Here E. R. $B.$ $\nu(x, \delta; \Lambda)$ integral and E. R. $B.$ $\nu'(x, \delta; \Lambda)$ integral are understood by the meaning of [2] p. 377 Def. 1. and $h(x, x') \equiv \tilde{f}(x) \cdot \tilde{g}(x')$ satisfies the property $\tilde{f}(\Lambda x) \cdot \tilde{g}(\Lambda' x') \equiv \tilde{f}(x) \cdot \tilde{g}(x')$ for any $\Lambda\Lambda'$.

Suppose that for each space-like pair of points (x, x'), $0 \leqq \sigma < l$ (namely $\delta(\vec{x}) \cdot \delta(\vec{x}')$) is used except for small probability, and for each not space-like pair of points (x, x'), $\sigma = l$ (namely $f(\vec{x}) \cdot g(\vec{x}')$) is used. Since space-like property is invariant by Lorentz transform, then causality condition is satisfied independently of Lorentz transform, but divergences related to commutation relations cannot be seen in relativistic frame. These assumptions seem to be effectively given by the mean $\int d(\Delta(x_0))$ in definition 4. Here the use of $\delta(\vec{x})\cdot\delta(\vec{x}')$ can be understood that the effective ranges are not one point but look alike to one point. This is also effective to the macrocausal interpretation of almost every process.

At last since this $\nu(x, \delta; \Delta)$ continue the change depending on the relativistic distance σ, then the structures of $C(n, +, \Delta)$ and $C(n, -, \Delta)$ (in space-like position) are also changed depending on this, and then it seems that this situation is like to the radiation and absorption of photon etc.

References

[1] H. Yamagata: The causality condition in nowhere dense perfect model. Proc. Japan Acad., **41** (1), 73–78 (1965).

[2] ——: Singular cut-off process and Lorentz covariance. Ibid., **41** (5), 377–382 (1965).

[3] ——: Singular cut-off process and Lorentz properties. Ibid., **41** (7), 547–552 (1965).

[4] M. A. Denjoy: Sur l'integration riemanienne. Comptes rendus. **169**, 219–220 (1919).

[5] И. Л. Бонди: Научные сообщения и задачи А-интеграл и обобщенные функции. Успехи Матем. Наук., т. **XIX**, Вып. 2 (116), 131–138 (1964).

[6] K. Kunugui: Sur une generalization de l'integrale. Fundamental and Applied Aspects of Mathematics, pp. 1–30 (1959).

[7] L. Schwartz: Theorie des distribution. I, II. Hermann, Paris (1950, 1951).

[8] M. Dresden: Aspects of abstruct field theory. Lectures in theoretical physics Vol. **II**. Interscience Publishers Inc., New York, pp. 366–434 (1959).

[9] S. Tomonaga: On a relativistically invariant formulation of the quantum theory of wave fields. Prog. Theor. Phys., **1**., 27–42 (1946).

[10] H. Yukawa: Quantum theory of non-local fields. Part 1. free field. Phys. Rev., **77** (2), 219–226 (1950).