29. On Differential Operators with Real Characteristics

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1. In the recent note [2] we constructed a wave operator of the form

$$(1)\qquad L=\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}-f\frac{\partial}{\partial t}-g,$$

where f and g are real valued infinitely differentiable functions in R^3, for which the local uniqueness of the Cauchy problem does not hold, if we give initial values on any domain on $S=\{(t, x, y);\ x^2+y^2=1\}$ and solve outward from S. A. Pliś [3], using the example of L. Hörmander [1], pp. 225, gave an example of differential equations possessing solutions with arbitrarily small supports.

In this note, by the method of A. Pliś [3], we prove the following:

Theorem. *Let $Q(\partial/\partial x)=Q(\partial/\partial x_1, \partial/\partial x_2)$ be a homogeneous differential operator of order $m(\geqq 1)$ in R^2 and of the form*

$$(2)\qquad Q\Big(\frac{\partial}{\partial x}\Big)=\sum_{j+k=m} a_{jk}\frac{\partial^m}{\partial x_1^j\partial x_2^k}.$$

Assume that there exist a real vector $N=(N_1, N_2)\neq 0$ such that $Q(N_1, N_2)=0$. Then there exist complex valued infinitely differentiable functions $b_{jk}(x)$ $(j+k\leqq m)$ which vanish at the origin with all their derivatives, and the local uniqueness of the Cauchy problem for the operator

$$(3)\qquad P\Big(x, \frac{\partial}{\partial x}\Big)=Q\Big(\frac{\partial}{\partial x}\Big)+\sum_{j+k\leqq m} b_{jk}(x)\frac{\partial^{j+k}}{\partial x_1^j\partial x_2^k}$$

does not hold for any smooth curve $\varphi(x)=0$ if $\varphi(0)=0$ and grad $\varphi(0)\neq 0$.

Remark. We shall prove that, for any $\varepsilon>0$, there exists a solution $u_\varepsilon(x)$ satisfying the equation $P(x, \partial/\partial x)u_\varepsilon(x)=0$ such that

$$(0, 0)\in \operatorname{supp} u_\varepsilon{}^{1)}\subset\{x;\ \varphi(x)\geqq 0,\ x_1^2+x_2^2<\varepsilon^2\}.$$

This means that, for any domain Ω containing the origin, we can not give any boundary condition on the boundary of Ω such that we may determine a unique solution of $P(x, \partial/\partial x)u_\varepsilon=0$.

Corollary. *Let $M(\partial/\partial x)=M(\partial/\partial x_1, \cdots, \partial/\partial x_\nu)$ be a homogeneous differential operator of order $m(\geqq 1)$ in $R^\nu(\nu\geqq 3)$. Assume that*

1) For a function $u(x)$, supp $u=$ the closure of $\{x;\ u(x)\neq 0\}$.

there exist non-zero real vectors $\xi'=(\xi'_1,\cdots,\xi'_\nu)$ *and* $\xi''=(\xi''_1,\cdots,\xi''_\nu)$ *such that* $M(\xi')\neq 0$ *and* $M(\xi'')=0$. *Then, there exist complex valued functions* $B_{j_1\cdots j_\nu}(y_1,y_2)\in C^\infty(R^2)$, $j_1+\cdots+j_\nu\leqq m$, *whose derivatives all vanish at the origin, and for the operator*

$$L\Big(x,\frac{\partial}{\partial x}\Big)=M\Big(\frac{\partial}{\partial x}\Big)+\sum_{j_1+\cdots+j_\nu\leqq m}B_{j_1\cdots j_\nu}(x\cdot\xi',x\cdot\xi'')^{2)}\frac{\partial^{j_1+\cdots+j_\nu}}{\partial x_1^{j_1}\cdots\partial x_\nu^{j_\nu}},$$

the local uniqueness of the Cauchy problem does not hold for any surface $\{x;\varphi(x\cdot\xi',x\cdot\xi'')=0\}$, *where* $\varphi(y_1,y_2)$ *is of class* $C^1(R^2)$ *such as* $\varphi(0,0)=0$, *grad* $\varphi(0,0)\neq 0$.

2. Proof of Theorem. First we prove a lemma with a little modified form of A. Pliś [3].

Lemma (*A. Pliś*). *There exists a complex valued function* $f(t,y)$ *in* $C^\infty(R^2)$, *whose derivatives all vanish at the origin, such that, for any* $\varepsilon>0$, *and smooth function* $\psi(t,y)$ *such as* $\psi(0,0)=0$ *and grad* $\psi(0,0)\neq 0$, *we have a solution* $w_\varepsilon(t,y)$ *of the equation*

$$\frac{\partial}{\partial t}w(t,y)=f(t,y)\frac{\partial}{\partial y}w(t,y) \tag{4}$$

whose support contains the origin and is contained in

$$\{(t,y);\psi(t,y)\geqq 0,\ t^2+y^2<\varepsilon^2\}.$$

Proof of Lemma. We follow the method of A. Pliś [3]. L. Hörmander [1] constructed complex functions $u(t,y)$ and $a(t,y)$ of class $C^\infty(R^2)$ and vanishing for $t\leqq 0$, such that the equation $\partial/\partial t u(t,y)=a(t,y)\partial/\partial y u(t,y)$ is satisfied and supp $u=\{(t,y);t\geqq 0\}$. Setting

$$v(\tau,\theta)=u(\tau-\theta^2,\theta),\ b(\tau,\theta)=(1-2\theta a(\tau-\theta^2,\theta))^{-1}a(\tau-\theta^2,\theta),$$

we obtain an equation $\partial/\partial\tau v(\tau,\theta)=b(\tau,\theta)\partial/\partial\theta v(\tau,\theta)$. If $\tau<q$ for a constant $q>0$, we have $t+y^2=(\tau-\theta^2)+\theta^2<q$. Hence, for a sufficiently small fixed $q^0>0$, the complex functions $v(\tau,\theta)$ and $b(\tau,\theta)$ are of class C^∞ in $\{(\tau,\theta);\tau\leqq q^0\}$ and vanish for $\tau\leqq\theta^2$, and supp v contains the origin. Let $A(s)$ be a function of class $C^\infty(R)$ such that $0\leqq A(s)\leqq 1$, $A(s)=0$ for $|s|\geqq 1$ and $A(0)=1$. Consider the functions

$$\begin{aligned} &w(t,y;t^0,y^0;r)=v(rA((t-t^0)/r),y-y^0),\\ &c(t,y;t^0,y^0;r)=A'((t-t^0)/r)b(rA((t-t^0)/r),y-y^0)\end{aligned} \tag{5}$$

for $0<r<q^0$. Then, setting

$$R(t^0,y^0;r)=\{(t,y);|t-t^0|\leqq r,\ |x-x^0|\leqq r^{1/2}\},$$

we have

$$\phi^{3)}\neq \text{supp}\, w\subset R(t^0,y^0;r),\ \text{supp}\, c\subset R(t^0,y^0;r), \tag{6}$$

and w, c satisfy the equation

$$\frac{\partial}{\partial t}w(t,y;t^0,y^0;r)=c(t,y;t^0,y^0;r)\frac{\partial}{\partial y}w(t,y;t^0,y^0;r). \tag{7}$$

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2) For a real vector $\xi=(\xi_1,\cdots,\xi_\nu)$ and $x=(x_1,\cdots,x_\nu)\in R^\nu$, $x\cdot\xi$ denotes the inner product $x\cdot\xi=x_1\xi_1+\cdots+x_\nu\xi_\nu$.

3) ϕ denotes the empty set.

Since $v(\tau, \theta)$ and $b(\tau, \theta)$ vanish for $\tau \leqq \theta^2$, we have

$$\tau^{-M}\frac{\partial^{j+k}}{\partial\tau^j\partial\theta^k}v(\tau, \theta)\to 0,\ \tau^{-M}\frac{\partial^{j+k}}{\partial\tau^j\partial\theta^k}b(\tau, \theta)\to 0\ (\tau\searrow 0)$$

uniformly for any fixed j, k, and $M>0$. Hence, remarking $rA((t-t^0)/r)\leqq r$, we have by (5)

$$(8)\qquad \begin{aligned}&\frac{\partial^{j+k}}{\partial t^j\partial y^k}w(t, y; t^0, y^0; r)\to 0,\\ &\frac{\partial^{j+k}}{\partial t^j\partial y^k}c(t, y; t^0, y^0; r)\to 0\end{aligned}$$

when $r\to 0$, uniformly in R^2 for any fixed j and k. Now, we set

$$R_{1,n}=R(n^{-1}, 0; |n|^{-5}),\ R_{2,n}=R(0, n^{-1}; |n|^{-5}),\ (n=\pm 1, \pm 2, \cdots).$$

Then there exists a positive integer $n^0(\geqq q^{0-5})$ such that

$$(9)\qquad R_{j,n}\cap R_{j',n'}=\phi, \text{ if } |n|\geqq n^0, \text{ and } (j, n)\neq(j', n').$$

We set, for an integer $l(|l|\geqq n^0)$,

$$w_{1,l}=\sum_{n=l}^{\pm\infty}w(t, y; n^{-1}, 0; |n|^{-5}),\ w_{2,l}=\sum_{n=l}^{\pm\infty}w(t, y; 0, n^{-1}; |n|^{-5}),$$

if $l\gtrless 0$ respectively and set

$$f(t, y)=\sum_{|n|\geqq n^0}\{c(t, y; n^{-1}, 0; |n|^{-5})+c(t, y; 0, n^{-1}; |n|^{-5})\}.$$

Then, by (6)—(9), we have $w_{j,l}(j=1, 2, |l|\geqq n^0)$, $f(t, y)\in C_0^\infty(R^2)$,

$$(10)\qquad (0, 0)\in \operatorname{supp} w_{j,l}\subset\{(t, y);\ t^2+y^2<4|l|^{-2}\}$$

and every $w_{j,l}(t, y)$ satisfies the equation (4).

Now, let $\psi(t, y)$ be a function of class C^1 in a neighborhood of the origin such that $\psi(0, 0)=0$ and $\operatorname{grad}\psi(0, 0)\neq 0$. Then

$$\psi(t, y)=\alpha t+\beta y+o(\sqrt{t^2+y^2}) \text{ where } (\alpha, \beta)\neq 0.$$

Hence, for any $\varepsilon>0$, we can select $j(=1$ or $2)$ and integer $l(|l|\geqq \operatorname{Max}\{n^0, 2\varepsilon^{-1}\})$ such that

$$(0, 0)\in \operatorname{supp} w_{j,l}\subset\{(t, y);\ \psi(t, y)\geqq 0,\ t^2+y^2<4|l|^{-2}\leqq\varepsilon^2\}.$$

This completes the proof. Q.E.D.

Proof of Theorem. Take a real vector $\xi^0=(\xi_1^0, \xi_2^0)\neq 0$ such that $Q(\xi_1^0, \xi_2^0)\neq 0$, then ξ^0 and N are linearly independent. If we transform the coordinates (x_1, x_2) to (t, y) by the non-singular transformation: $t=\xi_1^0x_1+\xi_2^0x_2$, $y=N_1x_1+N_2x_2$, then the differential polynomial $Q(\xi_1, \xi_2)$ is transformed to $Q'(\lambda, \eta)=Q(\xi_1^0\lambda+N_1\eta, \xi_2^0\lambda+N_2\eta)$ where (λ, η) corresponds to the differentiations $(\partial/\partial t, \partial/\partial y)$. Hence we have $Q'(1, 0)=Q(\xi_1^0, \xi_2^0)\neq 0$ and $Q'(0, 1)=Q(N_1, N_2)=0$, consequently we can write $Q'(\alpha, \eta)=Q_0'(\lambda, \eta)\lambda$ where $Q_0'(\lambda, \eta)$ is differential polynomial homogeneous of order $m-1$. Set $P'(t, y, \partial/\partial t, \partial/\partial y)=Q_0'(\partial/\partial t, \partial/\partial x)(\partial/\partial t-f(t, y)\partial/\partial y)$ with the function constructed in Lemma. Then all the solutions $w(t,y)$ of the equation (4) necessarily satisfy the equation $P'(t, y, \partial/\partial t, \partial/\partial y)w(t, y)=0$. Consequently we see that the local uniqueness of the Cauchy problem for the operator P' does not hold for any curve $\psi(t, y)=0$ of Lemma. If we re-transform the coordinates (t, y) to (x_1, x_2), we can easily see that $P'(t, y, \partial/\partial t, \partial/\partial y)$ is transformed to $P(x, \partial/\partial x)$ of the form (3) such as $b_{jk}(x)(j+k\leqq m)$ satisfy the conditions of Theorem, and that $\psi(t, y)$ is transformed to $\varphi(x_1, x_2)$ with one to one correspondence.

Q.E.D.

Proof of Corollary. We can linearly transform the coordinates $(x_1, \cdots, x_\nu)$ to $(t, y_1, \cdots, y_{\nu-1})$ such that the transformed differential polynomial $M'(\lambda, \eta_1, \cdots, \eta_{\nu-1})$ satisfies the conditions $M'(1, 0, \cdots, 0)\neq 0$, $M'(0, 1, 0, \cdots, 0)=0$, and the planes $x\cdot\xi'=0$, $x\cdot\xi''=0$ are transformed to the planes $t=0$ and $y_1=0$ respectively. Set $Q'(\partial/\partial t, \partial/\partial y_1)=M'(\partial/\partial t, \partial/\partial y_1, 0, \cdots, 0)$, then we can write $Q'(\partial/\partial t, \partial/\partial y_1)=Q_0'(\partial/\partial t, \partial/\partial y_1)\partial/\partial t$ where $Q_0'(\partial/\partial t, \partial/\partial y_1)$ is a homogeneous differential polynomial of order $m-1$. Next, with a function f defined in Lemma, we set $P'(t, y_1, \partial/\partial t, \partial/\partial y_1)=Q_0'(\partial/\partial t, \partial/\partial y_1)(\partial/\partial t-f(t,y_1)\partial/\partial y_1)$. Then, for the operator P', the local uniqueness of the Cauchy problem does not hold for any curve $\psi(t, y_1)=0$ satisfying the condition of Lemma. Considering $L'(t, y, \partial/\partial t, \partial/\partial y)\equiv P'(t, y_1, \partial/\partial t, \partial/\partial y_1)$, we can easily see that, for the operator L', the local uniqueness does not hold for any surface $\{(t, y); \psi(t, y_1)=0\}$ with the function ψ defined in the proof of Lemma, since the solution $w(t, y_1)$ of $P'w=0$ is also the solution of $L'w=0$ by considering as a function in R^ν. Consequently, re-transforming the coordinates $(t, y_1, \cdots, y_{\nu-1})$ to $(x_1, \cdots, x_\nu)$, we get the desired operator $L(x, \partial/\partial x)$. Q.E.D.

References

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