23. *Two-Dimensional Diffraction of Acoustic and Electromagnetic Waves by an Open Boundary*

By Yoshio HAYASHI

Department of Mathematics, College of Science and Engineering, Nihon University, Tokyo

(Comm. by Kinjirô KUNUGI, M.J.A., Feb. 12, 1966)

1. It is well known that the analyses of two-dimensional acoustic and electromagnetic fields are reduced to that of the Helmholtz equation

$$\Delta u + k^2 u = 0, \tag{1}$$

where u is a velocity potential in the case of acoustics and is a z-component of electric or magnetic fields in the case of electromagnetism. $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, k^2 $(Im.\ k \leqq 0)$ is a constant, and a factor $e^{\gamma z + i\omega t}$ is supressed throughout. The boundary conditions

$$u = 0, \tag{2}$$

or

$$\frac{\partial u}{\partial n} = 0 \tag{3}$$

is prescribed on a given boundary L, where n is a given unit normal on L. Further, u is required to satisfy the radiation condition at infinity.

When L is a closed contour, the above mentioned problems have long been a subject of many investigations, and (1) has been solved by various techniques. In fact, when L is of particular geometry, say a whole circle or a whole straight line, (1) has been solved by means of a Fourier-series or a Fourier transform technique, and when L is of semi-infinite extent, it has been solved by a Wiener-Hopf technique [1]. When L is a closed contour of a general geometry, then (1) has been converted into a second kind integral equation of Fredholm over L, which may be solved by a conventional way.

However, when L is open, that is, when L is an open arc or a union of open arcs, say a circle or a line with arbitrary slots, no rigorous and general analysis has been studied on (1), and the problems have usually been solved approximately assuming that there is only one narrow slit in the boundary and that the distribution of field components in the slit is known [2], [3]. Since we find a lot of counterparts of the problems in practical fields, for example, in a theory of a slotted antenna and a leaky waveguide, a mathematical theory of the problems is worth studying, while the theory itself is of interest from a mathematical point of view. So, it is the purpose of this paper to show how to solve (1) rigorously and generally when L is a union of arbitrary number of arbitrary (but smooth) arcs.

2. To begin with, the boundary condition (2) will be considered.

Suppose that $L=\sum_{j=1}^{\nu} L_j$ where L_j is a piecewise smooth, non-intersecting open arc. Suppose that $C_j=C_j^+ + C_j^- + C_j^*$ is a closed contour which encloses L_j in it, where respectively, C_j^+ and C_j^- are placed close to L in plus and minus sides of L with respect to n, and C_j^* is circular arcs of small radius surrunding the end points of L_j. Then, with help of the Green formula applied to a domain bounded by $C=\sum_{j=1}^{\nu} C_j$, $u(x)$ at a point x in the domain is represented by a line integral over C, which is reduced, in the limit as $C^\pm$ tend to L and when the radius of C_j^* tends to zero, to

$$u(x)=\int_L \tau(y)H_0(k\,|\,x-y\,|)\,ds_y-u^*(x), \qquad x \notin L, \tag{4}$$

by virtue of (2), under the assumption that the integrals over all of C_j^* vanish. In (4), y is a point on L, $H_0(k\,|\,x-y\,|)$ is the zeroth order Hankel function of the second kind, $|\,x-y\,|$ is the distance between x and y, $u^*(x)$ is a given primary field, and τ is a unknown function which is the difference of the limiting values of the normal derivatives of u in the negative and positive sides of L. Again by virtue of (2), (4) is reduced to

$$\int_L \tau(y)H_0(k\,|\,x-y\,|)ds_y=u^*(x), \qquad x \in L \tag{5}$$

as $x \notin L$ tends to a point x on L, which is a Fredholm integral equation of the first kind, the kernel of which has a weak singularity.

As mentioned above, we have assumed, when (4) was derived, that integrals over small circles C_j^* around end points of L disappear. We can prove that this is equivalent to take the edge conditions at the end points c_j $(j=1, 2, \cdots, 2\nu)$ into account; (6) $\tau(y)=o(|\,y-c_j\,|^{-1/2})$. It is also proved that this condition is equivalent to the conventional one derived from the requirement of finite energy at c_j.

Conversely, it is proved that if u is defined by (4) by the substitution of a solution τ of (5) which satisfies (6), then u is the exact solution of the original problem, that is, the u satisfies (1) when $x \notin L$, (2) when $x \in L$, (6) when $x \to c_j$ and the radiation condition at infinity. Therefore, (5) is our fundamental equation, the solution of which will be obtained in section 4.

3. Next, the boundary condition (3) will be considered. Suppose that L^c is a union of arbitrarily chosen ν arcs; $L^c=\sum_{j=1}^{\nu} L_j^c$, such that $C=L+L^c$ is a piecewise smooth, non-intersecting, closed contour.

Let n be a unit normal on C and let S^- and S^+ be domains bounded by C which exist in negative and positive sides of C respectively with respect to n. Assume that, $G^+(x, y)$ and $G^-(x, y)$ are functions of x and y which are regular in S^+ and S^- respectively and satisfies (1) there when $x \neq y$, $G^\pm(x, y) \to -\frac{1}{2} \log |x-y|$ when $x \to y$, and

$$\frac{\partial}{\partial n_y} G^\pm(x, y) = 0$$

when $x \neq y$ and $y \in C$.

Then, the Green formula applied to u and $G^\pm$ in $S^\pm$, togetherwith the boundary condition (3), leads to integral representations of $u(x)$, respectively, for $x \in S^+$ and for $x \in S^-$, which are

$$(7) \qquad u(x) = -\int_{L^C} \tau(y) G^+(x, y) ds_y - u^{*+}(x), \qquad x \in S^+,$$

$$u(x) = \int_{L^C} \tau(y) G^-(x, y) ds_y - u^{*-}(x), \qquad x \in S^-,$$

where $\tau(y)$ is the values of $\partial u/\partial n$ on L^c, and $u^{*\pm}(x)$ are given primary fields in $S^\pm$. Note that u and $\partial u/\partial n$ in $S^\pm$ should be continuous when they traverse L^c; (8) $u^+ = u^-$, $(\partial u/\partial n)^+ = (\partial u/\partial n)^-$ on L^c.

An integral equation for the unknown function τ in (7) is derived by tending $x \in S^\pm$ to a point $x \in L^c$ in (7) and by taking conditions (8) into account, which is

$$(9) \qquad \int_{L^C} \tau(y) \{G^+(x, y) + G^-(x, y)\} ds_y = u^{*-}(x) - u^{*+}(x), \qquad x \in L^c.$$

Converself, if u is defined by (7) with a solution τ of (9), then we can prove that u is the required solution, that is, it satisfies all of (1), (3), (6), (8) and the radiation condition at infinity. Therefore (9) is the fundamental equation for the original problem in this case. Note that (9) is essentially the same as (5), because the singularity of the kernel of (9) is the same as that of (5).

4. The fundamental equation in cases of (2) and (3), which are formally the same and are equivalent to

$$(10) \quad \frac{2}{\pi i} \int_L \tau(y) \log k|x-y| ds_y = u^*(x) - \int_L \tau(y) h(k|x-y|) ds_y, \quad x \in L,$$

where h is a given kernel, the second derivative of which is integrable over L, and where $u^*(x)$ is a given analytic function, is considered in the following.

On taking the tangential derivative with respect to x, (10) is transformed to

$$(11) \qquad \frac{1}{\pi i} \int_L \frac{\delta(\zeta)}{\zeta - z} d\zeta = f(z),$$

where z and ζ are complex variables corresponding to points x and y respectively, $\sigma(z) = \dfrac{\tau(z)}{z'}$ and $f(z)$ is

(12)

$$f(z)=f(s)=\frac{-1}{2z'(s)}\frac{\partial u^*}{\partial s}+\frac{1}{z'(s)}\int_L \tau(s')\frac{\partial}{\partial s}\left\{\frac{-1}{\pi}\hat{\theta}(s,s')+\frac{1}{2}h(s,s')\right\}ds'.$$

In (12), s and s' are arc lengths on C corresponding to points x and y respectively, $\hat{\theta}(s,s')$ is an angle sustained by the vector $\overrightarrow{xy}$ with respect to a fixed direction, say the x-axis in a xy-plane, and $h(s,s')$ is the function of s and s' corresponding to $h(k\,|\,x-y\,|)$.

It is known [4], [5], that a solution of (11) is

$$\sigma(z)=X(z)\left\{\frac{1}{\pi i}\int_L \frac{1}{\zeta-z}\frac{f(\zeta)}{X(\zeta)}d\zeta+\sum_{n=0}^{\nu-1}p_n z^n\right\}, \tag{13}$$

where the notations are the same as those in [5]. When transformed back to the original variables x and y, (13) is

$$\frac{\tau(s)}{z'(s)X(s)}-\int_L K(s,s')\tau(s')ds'=g(s), \tag{14}$$

where

$$K(s,s')=\frac{1}{\pi i}\int_L \frac{1}{X(t)\{z(s)-z(t)\}}\frac{\partial}{\partial t}\left\{\frac{-1}{\pi}\hat{\theta}(t,s')+\frac{1}{2}k(t,s')\right\}dt. \tag{15}$$

$$g(s)=\frac{-1}{2\pi i}\int_L \frac{1}{X(t)\{z(t)-z(s)\}}\frac{\partial u^t}{\partial t}dt+\sum_{n=0}^{\nu-1}p_n z^n(s).$$

(14) is a second kind integral equation of Fredholm, the kernel $K(s,s')$ of which is shown to be $\iint_L |K(s,s')|^2 ds ds'<\infty$, hence it is solvable by a conventional way.

In cases where $h(k\,|\,x-y\,|)=h(s,s')=h(s-s')$, (10) is solved directly without refering to (14). Actually, when L is composed of circular arcs, (10) is shown to reduce to the fundamental equation in [5], where it has been solved directly by the generalized formula of (13). Results similar to that in [5] will be shown in the following papers for the cases of a grating of plane strips, etc. The results in this paper, which is a résumé of the work, will appear, in its full text, in some journal later.

References

[1] Nobel, B.: Methods based on the Wiener-Hopf Technique. Pergamon Press, N. Y. (1958).

[2] Morse, P. M., and H. Feshbach: Methods of Theoretical Physics. McGraw-Hill, N. Y. (1953).

[3] Wait, J. R.: Electromagnetic Radiation from Cylindrical Structure. Pergamon Press, N. Y. (1959).

[4] Muskhelishvili, N. I.: Singular Integral Equations. P. Noordhoff N. V. Groningen (1953).

[5] Hayashi, Y.: Electromagnetic field in a domain bounded by coaxial circular cylinders with slots. Proc. Japan Acad., **40** (5) (1964); Jour. Applied Scientific Research (to appear).