22. A Duality Theorem for Locally Compact Groups. III

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1. In the previous paper [1], we proved a duality theorem for $G=SL(2, R)$ as follows.

Let Ω_0 be the set of all equivalence classes of irreducible unitary representations of G, and $D=\{U_g^D, \mathfrak{H}^D\}$ be a representative of each element in Ω_0. We call an operator field $\boldsymbol{T}_0=\{T_0(D)\}$ over Ω_0, *admissible* when

(1) $T_0(D)$ is a unitary operator in $\mathfrak{H}^D$ for any D in Ω_0.

(2) For any irreducible decomposition $\int D^\lambda d\nu(\lambda)$ of $D_1 \otimes D_2$ which is related by U,

$$U(T_0(D_1)\otimes T_0(D_2))U^{-1}=\int T_0(D^\lambda)d\nu(\lambda).$$

Under these definitions, the main result of [1] is as follows.

Proposition. *For any admissible operator field $\boldsymbol{T}_0$, there exists unique element g in G such that $T_0(D)=U_g^D$ for any D in Ω_0.*

The purpose of this article is to prove the same result for connected semisimple Lie group G with finite centre.

Concerning to [3] and proof in this article, we can deduce easily,

Corollary. *For connected semisimple Lie group G with finite centre and without compact factor, there exist finite irreducible unitary representations $\{D_j\}$. And the assumption* (1) *about unitarity of $T_0(D)$ is replaced by weaker assumption,*

(1′) *$T_0(D_j)$ is a non-singular bounded operator in $\mathfrak{H}^{D_j}$, and $T(D)$is a closed operator in $\mathfrak{H}^D$ for any D in Ω_0.*

2. Proof of the proposition. Let G be a connected semisimple Lie group with finite centre. In [5], Harish-Chandra showed such a G is type I. So any irreducible unitary representation of G is given as an outer tensor product of irreducible unitary representations of simple groups which are factors of G. Then it is sufficient to prove the proposition for such simple Lie groups.

For compact groups, there exists Tannaka's result [6], which assures the same proposition in this case. Hence, hereafter we consider only a non-compact connected simple Lie group G with finite centre $Z(G)$.

While in general, let $R\sim R'=\int D^{\lambda}d\sigma(\lambda)$ be an irreducible decomposition of the regular representation of type I group G, this decomposition is unique up to unitary equivalence. Now, for given admissible operator field $\boldsymbol{T}_0=\{T_0(D)\}$, if we can prove the integrability of $\{T_0(D)\}$ with respect to σ, that is, that $\{T_0(D)v(D)\}$ is in $\mathfrak{H}^{R'}$ for any vector $\{v(D)\}$ in $\mathfrak{H}^{R'}$, then the unique extension $T_0(R)$ of $\boldsymbol{T}_0$ on $\mathfrak{H}^{R}$ is defined as an operator corresponding to $T_0(R')=\int T_0(D^{\lambda})d\sigma(\lambda)$. The unitary property of $\boldsymbol{T}_0$ leads the unitarity of $T_0(R)$ and uniformly boundedness of $\boldsymbol{T}_0$ over the components of R results the boundedness of $T_0(R)$.

It is easy to see $T_0(R)$ satisfies the conditions of the proposition 1 and lemma 2 in [2], so from the results of [2], we obtain the proof of the proposition immediately.

In the other hand, we consider Ω_0^F the family of equivalence classes of unitary representations of G which consists of subrepresentations of Kronecker products with finite multiplicity of elements of Ω_0 and their finite direct sums.

Lemma. *Ω_0^F contains the equivalence class of R, for non-compact connected simple Lie group G with finite centre.*

This lemma means that R is representable as a subrepresentation of $\sum_j\oplus(D_1^j\otimes\cdots\otimes D_n^j)$ $(D_k^j\in\Omega_0)$. So using the condition (2) of admissibility, $\{T_0(D)\}$ is integrable with respect to μ as a projection of $\sum_j\oplus(T_0(D_1^j)\otimes\cdots\otimes T_0(D_n^j))$. And if we replace the unitarity of $\boldsymbol{T}_0$ to the boundedness of these finite operators $\{T_0(D_k^j)\}$, the uniformly boundedness of $\boldsymbol{T}_0=\{T_0(D)\}$ over the components of R follows.

3. Proof of the lemma. F. Bruhat [7] gave a family of irreducible representations D of G, which are induced by representations τ of proper subgroup Γ of G, as follows.

i) According to Iwasawa decomposition KHN of G, put M the centralizer of H in K (M contains $Z(G)$). And put

$$\Gamma=MHN.$$

ii) For a character φ of H such that $\varphi^s\neq\varphi$ (for any Weyl transformation s), and any irreducible unitary representation σ of M, put $\tau(\gamma)=\tau(mhn)=\varphi(h)\sigma(m)$ for $\gamma=mhn$ in Γ, $m\in M$, $h\in H$, $n\in N$.

Applying the Frobenius' theorem on induced representations, to M and $Z(G)$, we can select finite τ_j $(1\leqq j\leqq t)$ such that the restriction of $\sum\oplus\tau_j$ to $Z(G)$ contains the regular representation of $Z(G)$ as a component. Consider $D_0=\sum_{j=1}^{t}\oplus D_j$, in which D_j is the induced representation by τ_j, then D_0 is induced representation of G by $\tau_0=\sum_{j=1}^{t}\oplus\tau_j$. And the restriction of multiple $\tau_0\otimes\cdots\otimes\tau_0$ to $Z(G)$ contains the regular representation of $Z(G)$.

We shall show that for any $t \geqq (2\dim \Gamma)+2$, t-multiple $D_0 \otimes \cdots \otimes D_0$ contains a subrepresentation which is equivalent to the regular representation R of G.

In fact, we apply Mackey's results ([8], Th. 12.1) which decompose products of induced representations, and get if two subgroups $\overbrace{\Gamma \times \cdots \times \Gamma}^{t}(=\Gamma^t)$ and $\widetilde{G}_t = \{(g, \cdots, g)\}$ in $\overbrace{G \times \cdots \times G}^{t}(=G^t)$ are regularly related, then the t-multiple $D_0 \otimes \cdots \otimes D_0$ is equivalent to

$$\int_{\Gamma^t G^t / \widetilde{G}_t} D(g_1(\tau_0) \otimes \cdots \otimes g_t(\tau_0); \Gamma^t(\hat{g})) d\nu(\hat{g}),$$

where $\hat{g} = (g_1, \cdots, g_t)$ runs over the set of representatives of these double cosets, and ν is a measure over $\Gamma^t \backslash G^t / \widetilde{G}_t$ such that a double cosetwise set in G^t is a null set with respect to the Haar measure μ^t of G^t if and only if its canonical image in $\Gamma^t \backslash G^t / \widetilde{G}_t$ is a ν-null set. $D(g_1(\tau_0) \otimes \cdots \otimes g_t(\tau_0); \Gamma^t(\hat{g}))$ shows the induced representation of G by the restriction of $g_1(\tau_0) \otimes \cdots \otimes g_t(\tau_0)$ $(g_j(\tau_0) = \{U^{\tau_0}(g_j \gamma_{g_j}'^{-1}), H^{\tau_0}\}$, a representation of the group $g_j^{-1}\Gamma g_j)$, to $\Gamma^t(\hat{g}) = g_1^{-1}\Gamma g_1 \cap \cdots \cap g_t^{-1}\Gamma g_t$.

Because of that $g_1(\tau_0) \otimes \cdots \otimes g_t(\tau_0)|_{Z(G)} \sim \tau_0 \otimes \cdots \otimes \tau_0|_{Z(G)}$ contains the regular representation of $Z(G)$, the proof is reduced to show the following,

1) Γ^t, $\widetilde{G}_t$ are regularly related in G^t.

2) For $t \geqq (2\dim \Gamma)+2$, the set $F = \{\hat{g}^t : \Gamma^t(\hat{g}^t) = Z(G)\}$ is μ^t-measure positive in G^t.

The proof of 1) is given in the previous paper [4], so we shall prove that 2) is true.

4. At first, we consider $G \sim \widetilde{G}_l$ as a transformation group over $\Gamma^l \backslash G^l$, then the isotropy group of coset m containing $(\hat{g}^l) = (g_1, \cdots, g_l)$ is $\Gamma^l(\hat{g}^l)$. As shown in [9] (p. 135), $\dim \Gamma^l(\hat{g}^l)$ is a upper semicontinuous function over G^l, so $E = \{\hat{g} : \dim \Gamma^l(\hat{g}^l) = 0\}$ is an open set in G^l. E is non-empty for $l = (\dim \Gamma)+1$. In fact, for arbitrarily given two proper closed subgroups K_1, K_2 in G, let $N(K_1, K_2) = \{g \in G : \dim K_1 = \dim (K_1 \cap g^{-1}K_2 g)\} = \{g \in G : \mathfrak{k}_1 \subset (\mathrm{ad}\, g)\mathfrak{k}_2\}$, where $\mathfrak{k}_j$ is the Lie algebra of K_j. Obviously $N(K_1, K_2)$ is closed, and the simplicity of G assures $N(K_1, K_2) \neq G$. That is, $G_0(K_1, K_2) = G - N(K_1, K_2)$ is a non-empty open set in G, and for any g in $G_0(K_1, K_2)$, $\dim (K_1 \cap g^{-1}K_2 g) < \dim K_1$. We take g_1 in G, and next g_2 in $G_0(g_1^{-1}\Gamma g_1, \Gamma)$, g_3 in $G_0(g_1^{-1}\Gamma g_1 \cap g_2^{-1}\Gamma g_2, \Gamma)$ and so on, finally we get $\hat{g}^l = (g_1, \cdots, g_l)$ in G^l for $l = (\dim \Gamma)+1$, and $\dim \Gamma^l(\hat{g}^l) = 0$. That is $E \neq \phi$.

We put $E' = \{\hat{g}^{l-1} = (g_1, \cdots, g_{l-1}) : (g_1, \cdots, g_{l-1}, e) \in E\}$ in G^{l-1}, then E' is a non-empty open set and $\dim (\Gamma^{l-1}(\hat{g}^{l-1}) \cap \Gamma) = 0$ for any $\hat{g}^{l-1} \in E'$. Especially, $\mu^{2l-1}(\{\hat{g}^{2l-1} = (\hat{g}^l, \hat{g}^{l-1}) : \hat{g}^l \in E, \hat{g}^{l-1} \in E'\}) \neq 0$.

Next we consider $N(g_1, g_2) = \{g \in G : g_1 = g^{-1}g_2 g\}$, for g_1, g_2 in G, then $N(g_1, g_2) = g_0\mathfrak{C}(g_1)$ for any g_0 in $N(g_1, g_2)$ and the centralizer

$\mathfrak{C}(g_1)$ of g_1. I.e., if one of g_1, g_2 is not in $Z(G)$, then $\mu(N(g_1, g_2))=0$.

For discrete (therefore countable) subgroup $\Gamma^l(\hat{g}^l)$, and $\Gamma^{l-1}(\hat{g}^{l-1}) \cap \Gamma$, $\hat{g}^l \in E$, $\hat{g}^{l-1} \in E'$, the set $N_0=\{g \in G: \Gamma^{2l}(\hat{g}^{2l}) \neq Z(G), \hat{g}^{2l}=(\hat{g}^l, \hat{g}^{l-1}g, g)\}$ is covered by countable sum of $N(g_1, g_2)$, in which $g_1 \in \Gamma^l(\hat{g}^l)-Z(G)$, $g_2 \in \Gamma^{l-1}(\hat{g}^{l-1}) \cap \Gamma - Z(G)$, so $\mu(N_0)=0$. Consequently, for any $\hat{g}^l \in E$, and $\hat{g}^{l-1} \in E'$, $\Gamma^{2l}(\hat{g}^{2l})=Z(G)$, for almost all g.

It is easy to see that $F_1=\{\hat{g}^{2l}: \Gamma^{2l}(\hat{g}^{2l})=Z(G)\}$ is measurable in G^{2l}, therefore $F_1'=\{\hat{g}^{2l}=(g_1, \cdots, g_l, g_{l+1}g_{2l}^{-1}, \cdots, g_{2l-1}g_{2l}^{-1}, g_{2l}): (g_1, \cdots, g_{2l}) \in F_1\}$ is measurable too. And the above discussions conclude $\mu^{2l}(F_1') \neq 0$, so $\mu^{2l}(F_1) \neq 0$.

For $t>2l$, since $\Gamma^t(\hat{g}^t)=\Gamma^{2l}(\hat{g}^{2l}) \cap \Gamma^{t-2l}(\hat{g}^{t-2l})=Z(G) \cap \Gamma^{t-2l}(\hat{g}^{t-2l})=Z(G)$, where $\hat{g}^t=(\hat{g}^{2l}, \hat{g}^{t-2l})$, $\hat{g}^{2l} \in F_1$, $\hat{g}^{t-2l} \in G^{t-2l}$, the result is immediate.

The corollary follows from the boundedness of $T(R)$ based on the boundedness of $\{T(D_j)\}$ and proposition in [3].

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