21. *Regularity of Orbits Space on Semisimple Lie Groups*

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1. Let G be a semisimple Lie group, and KHN be its Iwasawa decomposition, M be the subgroup $K \cap \mathfrak{C}(H)$ where $\mathfrak{C}(H)$ shows the centralizer of H.

F. Bruhat [1] shows that $\Gamma = MHN$ is a closed subgroup of G, and G is a disjoint sum of finite Γ-Γ double cosets which correspond to elements of Weyl group in one-to-one way.

While denote by $G^t = G \times \cdots \times G$ the direct product of G with multiplicity t and by $\widetilde{G}_t = \{(g, \cdots, g) \in G^t\}$ the diagonal subgroup of G^t, which is isomorphic to G.

There exists a question whether Γ^t and $\widetilde{G}_t$ are regularly related in G^t or not, in the sense of Mackey [2]. This problem is related to a problem of decomposability of Kronecker product of induced representations of G by representations of Γ, with multiplicity t (cf. [3]).

The purpose of this work is to solve this problem affirmatively.

Proposition. *Γ^t and $\widetilde{G}_t$ are regularly related in G^t.*

2. Proof of the proposition. At first, we can equate $\Gamma^t \backslash G^t / \widetilde{G}_t$ to $\Gamma^{t-1} \backslash G^{t-1} / \widetilde{\Gamma}_{t-1}$ by the map of representatives of cosets, $G^t \ni (g_1, g_2, \cdots, g_t) \rightarrow (g_1 g_t^{-1}, g_2 g_t^{-1}, \cdots, g_{t-1} g_t^{-1}) \in G^{t-1}$.

Using Glimms results [4], we can conclude that $\Gamma \backslash G / \Gamma$ is T_0 and the union of all lower dimensional Γ-Γ double cosets in G becomes a null set F in G, and $G' = G - F$ is open as a union of open cosets. Therefore it is sufficient to show the space $\Gamma^{t-1} \backslash (G')^{t-1} / \widetilde{\Gamma}_{t-1}$ is countably separated.

Again by [4], the last space is countably separated if and only if it is T_0. And for fixed l and closed subgroups $A \supset B$ in Γ^l, if $\Gamma^l \backslash (G')^l / A$ and $\hat{g} \Gamma^l \hat{g}^{-1} \cap A \backslash A / B$ are T_0 for any $\hat{g}$ in $(G')^l$, then $\Gamma^l \backslash (G')^l / B$ is T_0.

In this case, we put $A = \widetilde{\Gamma}_{l-1} \times \Gamma = \{(\gamma, \cdots, \gamma, \gamma') \in \Gamma^l\}$ and $B = \widetilde{\Gamma}_l$. Then easily we get, $\Gamma^l \backslash (G')^l / \widetilde{\Gamma}_{l-1} \times \Gamma \sim \Gamma^{l-1} \backslash (G')^{l-1} / \widetilde{\Gamma}_{l-1} \times \Gamma \backslash G' / \Gamma$ and $\hat{g} \Gamma^l \hat{g}^{-1} \cap A \backslash A / B \sim \Gamma^{l-1}(\hat{g}) \times \Gamma^1(g_l) \backslash \Gamma \times \Gamma / \widetilde{\Gamma}_2 \sim \Gamma^{l-1}(\hat{g}) \backslash \Gamma / \Gamma^1(g_l)$, where $\Gamma^{l-1}(\hat{g}) = \Gamma \cap g_1 \Gamma g_1^{-1} \cap g_2 \Gamma g_2^{-1} \cap \cdots \cap g_{l-1} \Gamma_{g_{l-1}}^{-1}$, and $\Gamma^1(g_l) = g_l \Gamma g_l^{-1} \cap \Gamma$, for $\hat{g} = (g_1, g_2, \cdots, g_l)$ in $(G')^l$. Consequently, if we prove $\Gamma^{l-1}(\hat{g}) \backslash \Gamma / \Gamma^1(g_l)$ is T_0, then by the induction with respect to l, we get the proof.

Now we shall show that $\Gamma^1(g)$ is conjugate to MH in Γ for any

g in G'. In fact, as shown in [1], for any g in G, there exists a $s(g)$ in $\Gamma g\Gamma$ such that $\Gamma^1(s(g))=MHN_{s(g)}$, where $N_{s(g)}=N\cap s(g)Ns(g)^{-1}$. But calculations of $\dim \Gamma^1(g)=\dim \Gamma^1(s(g))$ result if $\dim G=\dim \Gamma g\Gamma=\dim \Gamma s(g)\Gamma$ then $N_{s(g)}=\{e\}$, i.e. $\Gamma^1(s(g))=MH$, which is conjugate to $\Gamma^1(g)$ in Γ.

The space $\Gamma/\Gamma^1(g)\sim\Gamma/\Gamma^1(s(g))=\Gamma/MH$ is homeomorphic to N, therefore to its Lie algebra $\mathfrak{n}$ too. So $\Gamma^{l-1}(\hat{g})\backslash\Gamma/\Gamma^1(g_l)\sim\Gamma^{l-1}(\hat{g})\backslash\Gamma/MH$ is homeomorphic to the orbits space by the operations $\{\mathrm{ad}\,\gamma\}$ of adjoint representation restricted on $\mathfrak{n}$ for γ in $\Gamma^{l-1}(\hat{g})$ which is conjugate to a subgroup Γ' of MH in Γ. The general theory of Lie algebras gives, that $\mathfrak{n}$ is generated by roots vectors E_α of $\mathrm{ad}\,h$ such that

$$(\mathrm{ad}\,h)E_\alpha=e^{\alpha(\mathfrak{h})}E_\alpha, \quad \text{for} \quad h=\exp\mathfrak{h} \text{ in } H.$$

Since any m in M commutes with all h in H, $\{\mathrm{ad}\,m\}$ is a representation of compact group M with orthogonal matrices, which makes invariant the subspace $\mathfrak{n}_\lambda$ spanned by E_α's such that α's give a same real linear form λ on H. Let the projection of X in $\mathfrak{n}$ to $\mathfrak{n}_\lambda$ be X_λ, so $\Omega_J=\{X\colon X_{\lambda_j}=0,\ j\in J\}$ for some set J of indices is a closed subspace in $\mathfrak{n}$. It is enough to prove the proposition, we show that each orbit $\{(\mathrm{ad}\,\gamma)X\colon \gamma\in\Gamma'\}$ is closed in $\Omega_J-\bigcup_{J_1\supsetneqq J}\Omega_{J_1}$ which contains X.

For given $\hat{g}=(g_1,\cdots,g_l)$, we take $(s(g_j))(=s_j)$ as above, and let $g_j=\gamma_j s_j\gamma'_j$, $(\gamma_j,\gamma'_j\in\Gamma)$, and $\gamma_j=n_jh_jm_j$, $n_j\in N$, $h_j\in H$, $m_j\in M$. Then it is easy to see $\Gamma^{l-1}(\hat{g})$ is conjugate to

$$\begin{aligned}\Gamma'&=\Gamma\cap\gamma_1s_1\Gamma s_1^{-1}\gamma_1^{-1}\cap\cdots\cap\gamma_{l-2}s_{l-2}\Gamma s_{l-2}^{-1}\gamma_{l-2}^{-1}\cap s_{l-1}\Gamma s_{l-1}^{-1},\\&=\gamma_1(\Gamma\cap s_1\Gamma s_1^{-1})\gamma_1^{-1}\cap\cdots\cap\gamma_{l-2}(\Gamma\cap s_{l-2}\Gamma s_{l-2}^{-1})\gamma_{l-2}^{-1}\\&\qquad\cap(\Gamma\cap s_{l-1}\Gamma s_{l-1}^{-1})\\&=\gamma_1MH\gamma_1^{-1}\cap\cdots\cap\gamma_{l-2}MH\gamma_{l-2}^{-1}\cap MH\\&=n_1MHn_1^{-1}\cap\cdots\cap n_{l-2}MHn_{l-2}^{-1}\cap MH,\end{aligned}$$

in Γ. According to the uniqueness of decomposition $\Gamma=MHN$, we get that Γ' is commutator of $(n_1,n_2,\cdots,n_{l-2})$ in MH, that is, the subgroup of MH consists of $\gamma=mh$ such that $(\mathrm{ad}\,mh)X_j=X_j$ $(1\leqq j\leqq l-2)$ for $n_j=\exp X_j$. Let $X_j=\Sigma(X_j)_\lambda$, $(X_j)_\lambda\in\mathfrak{n}_\lambda$, then these relations are equivalent to $(\mathrm{ad}\,mh)(X_j)_\lambda=e^{\lambda(\mathfrak{h})}(\mathrm{ad}\,m)(X_j)_\lambda=(X_j)_\lambda$. While $(\mathrm{ad}\,m)$ is an orthogonal transformation and $\lambda(\mathfrak{h})$ is real, so $(\mathrm{ad}\,m)(X_j)_\lambda=(X_j)_\lambda$, and $e^{\lambda(\mathfrak{h})}(X_j)_\lambda=(X_j)_\lambda$. Consequently, Γ' is a direct product of a closed subgroup M_0 in M and a vector subgroup H_0 in H.

If a sequence $\{(\mathrm{ad}\,m_kh_k)X_1\colon m_k\in M_0,\ h_k\in H_0\}$ in a Γ'-orbit which is contained in some $\Omega_J-\bigcup_{J_1\supsetneqq J}\Omega_{J_1}$, converges to X_0 in the same space, then each component $(\mathrm{ad}\,m_kh_k)(X_1)_{\lambda_j}$ converges to $(X_0)_{\lambda_j}$, which are not zero for only $j\notin J$. This means their norms $e^{\lambda_j(\mathfrak{h}_k)}\|(X_1)_{\lambda_j}\|$ converges to non-zero finite value $\|(X_0)_{\lambda_j}\|$, and since H_0 is a vector subgroup of H, there exist a h_0 in H_0 such that $\|(\mathrm{ad}\,h_0)(X_1)_{\lambda_j}\|=\|(X_0)_{\lambda_j}\|$.

The compactness of M_0 assures the existence of subsequence of $\{(\operatorname{ad} m_k)X_1\}$ which converges to some $(\operatorname{ad} m_0)X_1$. That is, there is a subsequence converging to $(\operatorname{ad} m_0 h_0)X_1$ in this Γ'-orbit, obviously the limit of which must coincide to X_0. I.e., each orbit is closed in $\Omega_J - \bigcup_{J_1 \supsetneqq J} \Omega_{J_1}$. This completes the proof.

References

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