19. *A New Convergence Criterion of Fourier Series*

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§1. The object of this paper is to prove the following two theorems:

Theorem 1. *If* (i) f *is even,* (ii) $\int_0^t f(u)du=o(t)$ *as* $t\to 0$ *and* (iii) *for some* $\delta>0$, *there is an* $\eta(1>\eta>0)$ *such that*

$$\Theta(t)=\int_t^\delta |d\theta(u)|=O(t^{-\eta}) \quad \text{as} \quad t\to 0$$

where $\theta(u)=u^{-\eta}f(u)$, *then the Fourier series of* f *converges at the origin.*

Theorem 2. *If* f *is continuous and is of bounded variation and if there is an* $\eta>0$ *such that* (i) $t^{-\eta}\omega(t)>A>0$ *as* $t\to 0$ *and* (ii)

$$\Theta(t)=\int_t^\delta |d\theta(u)|=O(t^{-\eta}\omega(t)) \quad \text{as} \quad t\to 0$$

uniformly for all x, *where* $\theta(u)=u^{-\eta}\varphi_x(u)$, *then*

$$|s_n(x;f)-f(x)|\leqq A\omega(1/n) \quad \textit{for all} \quad x.$$

§2. Proof of Theorem 1. It is sufficient to prove that

$$s_n=\int_0^\delta f(t)\frac{\sin nt}{t}dt=o(1) \quad \text{as} \quad n\to\infty,$$

where δ is a fixed constant. We write $s_n=\int_0^{k/n}+\int_{k/n}^\delta=I_1+I_2$, where k is fixed but a large number. Then $I_1=o(1)$ as $n\to\infty$. By the assumption, $f(u)$ is of bounded variation on the interval $(k/n, \delta)$, and then $|I_2|\leqq V/n$, where V is the total variation of the function $f(t)/t$ on the interval $(k/n, \delta)$. Hence it is sufficient to show that $V=o(n)$. Since $f(t)/t=\theta(t)/t^{1-\eta}$, the required relation is that, for any given ε and a suitable k,

$$\int_{k/n}^\delta \left|d\left(\frac{\theta(t)}{t^{1-\eta}}\right)\right|\leqq \varepsilon n.$$

Now $d\Theta(t)=|d\theta(t)|$ and $|\theta(t)|=\left|\int_t^\delta d\theta(t)\right|\leqq\Theta(t)$ since we can suppose that $f(\delta)=0$ and then

$$\int_{k/n}^\delta \left|d\left(\frac{\theta(t)}{t^{1-\eta}}\right)\right|\leqq\int_{k/n}^\delta \frac{|d\theta(t)|}{t^{1-\eta}}+\int_{k/n}^\delta \frac{|\theta(t)|}{t^{2-\eta}}dt$$

$$\leqq\left[\frac{\Theta(t)}{t^{1-\eta}}\right]_{k/n}^\delta+A\int_{k/n}^\delta \frac{\Theta(t)}{t^{2-\eta}}dt\leqq A\frac{n}{k}<\varepsilon n.$$

This gives the required relation. Thus we get the theorem.

We have the following corollary which is a generalization of a theorem of Tomic [1].

Corollary 1. *If f is even, positive and continuous at the origin and there is an $\eta>0$ such that $t^{-\eta}f(t)$ is decreasing on the right neighbourhood of the origin, then the Fourier series of f converges at the origin.*

For, $\theta(t)=t^{-\eta}f(t)$ is decreasing and then

$$\Theta(t)=\theta(t)-\theta(\delta)=t^{-\eta}f(t)-\delta^{-\eta}f(\delta)=O(t^{-\eta}),$$

since f is bounded in the neighbourhood of the origin. Hence the condition (iii) is satisfied. Thus we get the theorem.

§ 3. Proof of Theorem 2. We can suppose that f is not constant, since the theorem is trivial when f is constant. We write

$$s_n(x;f)-f(x)=\frac{1}{\pi}\int_0^\pi \varphi_x(t)D_n(t)dt=\frac{1}{\pi}\left(\int_0^\delta+\int_\delta^\pi\right)$$
$$=\frac{1}{\pi}(I_1+I_2).$$

If $\omega(h_n)=o(h_n)$ as $h_n\to 0$, then f is constant, so that $1/n=O(\omega(1/n))$ as $n\to\infty$. Hence $I_2\leqq V/n=O(1/n)=O(\omega(1/n))$, where V is the total variation of $\varphi_x(t)/2\sin t/2$ over (δ,π). We write $I_1=\int_0^{1/n}+\int_{1/n}^\delta=I_{11}+I_{12}$, then

$$|I_{11}|\leqq\int_0^{1/n}|\varphi_x(t)D_n(t)|\,dt\leqq An\int_0^{1/n}|\varphi_x(t)|\,dt\leqq A\omega(1/n)$$

and

$$I_{12}=\int_{1/n}^\delta \frac{\varphi_x(t)}{2\sin t/2}\sin(n+1/2)tdt$$
$$=\int_{1/n}^\delta \frac{\varphi_x(t)}{t}\sin(n+1/2)tdt+\int_{1/n}^\delta \varphi_x(t)\left[\frac{1}{2\sin t/2}-\frac{1}{t}\right]\sin(n+1/2)tdt$$
$$=I_{121}+I_{122},$$

where $I_{122}=O(1/n)=O(\omega(1/n))$, since I_{122} is the n-th Fourier coefficient of bounded variation whose total variation is less that a constant. $|I_{121}|\leqq V_1/n$ where V_1 is the total variation of the function $\varphi_x(u)/u=\theta(u)/u^{1-\eta}$ over the interval $(1/n,\delta)$. It is sufficient to show that $V_1=O(n\omega(1/n))$. Now

$$\int_{1/n}^\delta\left|d\left(\frac{\theta(t)}{t^{1-\eta}}\right)\right|\int_{1/n}^\delta\frac{|d\theta(t)|}{t^{1-\eta}}+\int_{1/n}^\delta\frac{|\theta(t)|}{t^{2-\eta}}dt$$
$$\leqq\left[\frac{\Theta(t)}{t^{1-\eta}}\right]_{t=1/n}^\delta+\int_{1/n}^\delta\frac{\Theta(t)}{t^{2-\eta}}dt+\int_{1/n}^\delta\frac{|\theta(t)|}{t^{1-\eta}}dt$$
$$\leqq A+An\omega(1/n)+An^{1-\eta}\leqq An\omega(1/n)$$

which is the required. Thus we get the theorem.

As a special case, we get the following corollary which is a generalization of Tomic's theorem [1]:

Corollary 2. *If f is continuous and is of bounded variation and if, for any x, there is an $\eta>0$ such that $t^{\eta}\varphi_x(t)$ is positive, decreasing (or negative increasing) in the right neighbourhood of $t=0$, then* $|s_n(x;f)-f(x)|\leqq A\omega(1/n)$.

Reference

[1] M. Tomić: A convergence criterion for Fourier series. Proc. Amer. Math. Soc., 612-617 (1964).