18. An Extension of Certain Quasi-Measure

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1. Introduction. In 1956, I. M. Gelfand and A. M. Yaglom [3] pointed out importance of probability-theoretical treatments of certain partial differential equations. It would be interesting to construct (signed) measures on function spaces which stand in the same relation to some partial differential equations as the Brownian motion does to the heat equation. Let us consider the Cauchy problem for an equation:

$$\frac{\partial u}{\partial t}=-a\frac{\partial^4 u}{\partial x^4}+b\frac{\partial^2 u}{\partial x^2}\quad (a>0). \tag{1}$$

The solution u with an initial value f is given by

$$u(t,x)=\int_{-\infty}^{\infty} f(y)g(t,x-y)dy,$$

where

$$g(t,x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x-t(a\xi^4+b\xi^2)}d\xi.$$

Let Ω be a function space with a coordinate mapping x_t. It is natural to define a measure $\boldsymbol{P}_x$ of a cylinder set $C=\{\omega:(x_{t_1}(\omega), x_{t_2}(\omega),\cdots,x_{t_n}(\omega))\in\Gamma\}$ in Ω as follows;

$$\boldsymbol{P}_x[C]=\iint\cdots\int_\Gamma g(t_1,y_1-x)g(t_2-t_1,y_2-y_1)\cdots \times g(t_n-t_{n-1},y_n-y_{n-1})dy_1dy_2\cdots dy_n,$$

where $n\geqq 1$, $0\leqq t_1\leqq t_2\leqq\cdots\leqq t_n\leqq T$[1] and $\Gamma\subset R^n$. It is easy to see that $\boldsymbol{P}_x$ is well defined on the algebla $\mathfrak{F}$ consisting of all cylinder sets and that it is a finitely additive signed measure on $\mathfrak{F}$. We call $\boldsymbol{P}_x$ a *quasi-measure* corresponding to the equation (1).

Kolmogorov's extension theorem (cf. [5]) shows that if $\boldsymbol{P}_x$ is nonnegative, then $\boldsymbol{P}_x$ has the extension to the σ-algebra $\mathfrak{B}$ generated by $\mathfrak{F}$. But in our case it turns out that $\boldsymbol{P}_x$ may actually be negative and that its total variation is infinite. Therefore $\boldsymbol{P}_x$ can not be extended to $\mathfrak{B}$. This fact was shown in 1960 by V. Yu. Krylov [6] for a wider class of quasi-measures. At the present time we know some sufficient conditions in order that a quasi-measure may be extended to a σ-additive signed measure, which we will call a *Markovian system* (cf. [1], [7]).

In this note we try to obtain a reasonable extension of $\boldsymbol{P}_x$ to

1) Throughout this note, a positive constant T is fixed.

an algebra wider than $\mathfrak{F}$. Let us consider a system of equations:

$$\begin{cases} \dfrac{\partial u_1}{\partial t}=\alpha_1\dfrac{\partial^2 u_1}{\partial x^2}+\beta_1(u_2-u_1), \\ \dfrac{\partial u_2}{\partial t}=\alpha_2\dfrac{\partial^2 u_2}{\partial x^2}+\beta_2(u_1-u_2), \end{cases} \tag{2}$$

where $\alpha_1, \alpha_2 > 0$ and

$$\beta_1=\frac{(\alpha_1-b)\alpha_1\alpha_2}{a(\alpha_1-\alpha_2)}, \quad \beta_2=-\frac{(\alpha_2-b)\alpha_1\alpha_2}{a(\alpha_1-\alpha_2)}.$$

Eliminating u_2, we get

$$\frac{a}{\alpha_1\alpha_2}\frac{\partial^2 u_1}{\partial t^2}-a\Big(\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\Big)\frac{\partial^3 u_1}{\partial t\partial x^2}+\frac{\partial u_1}{\partial t}+a\frac{\partial^4 u_1}{\partial x^4}-b\frac{\partial^2 u_1}{\partial x^2}=0.$$

Formally, we obtain the equation (1) by sending off α_1 and α_2 to the infinity. On the other hand, it will be proved that there exists a Markovian system $\boldsymbol{P}^{(\alpha)}_{(j,x)}$ on the state space $\{1, 2\}\times R^1$ corresponding to (2). Thus the quasi-measure $\boldsymbol{P}_x$ is approximated by Markovian systems.

The procedure to approximate the equation (1) by the system (2) was suggested in the course of discussion with M. Nagasawa on an unpublished note of N. Ikeda, which gives a probabilistic treatment to the transport problem.

2. Construction of the Markovian system. In this section we construct a Markovian system corresponding to (2). We write

$$(p^{(\alpha)}_{jk}(t))=\exp\left\{t\begin{pmatrix} -\dfrac{\beta_1}{\alpha_1} & \dfrac{\beta_1}{\alpha_1} \\ \dfrac{\beta_2}{\alpha_2} & -\dfrac{\beta_2}{\alpha_2}\end{pmatrix}\right\}.$$

Because $\max_{j=1,2}\sum_{k=1}^{2}|p^{(\alpha)}_{jk}(t)| \leqq \exp\left\{2t\cdot\max_{j=1,2}\left|\dfrac{\beta_j}{\alpha_j}\right|\right\}\equiv e^{\gamma(\alpha)\cdot t}$, there exists a Markovian system $(\tilde{\theta}_t, \boldsymbol{P}^{(\alpha)}_j)$ with the space $\Omega_1=\{1, 2\}^{[0,\infty)}$ of elementary events, such that

$$\boldsymbol{P}^{(\alpha)}_j[\tilde{\theta}_t=k]=p^{(\alpha)}_{jk}(t) \qquad (1\leqq j, k\leqq 2).$$ [2)]

By $(\xi_t, \boldsymbol{P}_x)$, we denote the one-dimensional Brownian motion defined on the space Ω_2 of continuous paths with the generator $\dfrac{\partial^2}{\partial x^2}$.

Let $\tilde{z}_t=(\tilde{\theta}_t,\xi_t)$, $\tilde{\boldsymbol{P}}^{(\alpha)}_{(j,x)}=\boldsymbol{P}^{(\alpha)}_j\times\boldsymbol{P}_x$, and $\tilde{\Omega}=\Omega_1\times\Omega_2$. We get a Markovian system $(\tilde{z}_t, \tilde{\boldsymbol{P}}^{(\alpha)}_{(j,x)})$ defined on $\tilde{\Omega}$ with the state space $\{1, 2\}\times R^1$. As is easily seen, the Markovian system corresponds to a system of equations:

2) Theorem 1 in [7] holds not only for contraction semi-groups, but also for semi-groups T_t for which $\| T_t \| \leqq e^{\gamma t}$.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \dfrac{\partial^2 u_1}{\partial x^2} + \dfrac{\beta_1}{\alpha_1}(u_2 - u_1), \\ \dfrac{\partial u_2}{\partial t} = \dfrac{\partial^2 u_2}{\partial x^2} + \dfrac{\beta_2}{\alpha_2}(u_1 - u_2).^{3)} \end{cases}$$

Applying to $\tilde{z}_t$ Volkonskii's random time change (cf. [8]) induced by a positive additive functional $\int_0^t \alpha(\tilde{\theta}_s)^{-1} ds$,[4] we get a Markovian system $(z_t^{(\alpha)}, \tilde{\boldsymbol{P}}^{(\alpha)}_{(j,x)})$ which corresponds to (2). Let Ω be the set of all functions $\omega(t) = (\omega_1(t), \omega_2(t))$ in $[\{1, 2\} \times R^1]^{[0,\infty)}$ with the continuous second coordinate $\omega_2(t)$. Let $z_t(\omega) \equiv (\theta_t(\omega), x_t(\omega)) = \omega(t)$, and let

$$\boldsymbol{P}^{(\alpha)}_{(j,x)}[\omega : z_\cdot(\omega) \in A] = \tilde{\boldsymbol{P}}^{(\alpha)}_{(j,x)}[\tilde{z}^{(\alpha)}_\cdot \in A].$$

We obtain a signed measure $\boldsymbol{P}^{(\alpha)}_{(j,x)}$ on Ω. Thus we have

Lemma 1. *There exists a Markovian system $(z_t, \boldsymbol{P}^{(\alpha)}_{(j,x)})$[5] corresponding to (2), which is equivalent to $(\tilde{z}_t^{(\alpha)}, \tilde{\boldsymbol{P}}^{(\alpha)}_{(j,x)})$.*

It is essential for our argument that z_t does not depend on the parameter α, while $z_t^{(\alpha)}$ does. As for practical calculations, however, we utilize $z_t^{(\alpha)}$, because the construction of the latter is clearer than that of the former.

Let $\mathfrak{F}$ be the algebra consisting of all cylinder sets generated by x_t and let $\mathfrak{B}$ be the smallest σ-algebra including $\mathfrak{F}$.

Lemma 2. *We write*

$$B_1 = \{\omega : x_t(\omega) \text{ is differentiable at some } t\},$$

$$B_2 = \left\{\omega : \overline{\lim_{|t-s|\downarrow 0}} \frac{|x_t(\omega) - x_s(\omega)|}{\sqrt{|t-s| \cdot \lg |t-s|^{-1}}} = +\infty\right\},$$

$$B_3 = \left\{\omega : \overline{\lim_{t\downarrow s}} \frac{|x_t(\omega) - x_s(\omega)|}{\sqrt{|t-s| \cdot \lg \lg |t-s|^{-1}}} = +\infty \text{ for some } s\right\}.$$

Then $\boldsymbol{P}^{(\alpha)}_{(j,x)}[B] = 0$ for any $B \in \mathfrak{B}$ such that $B \subset B_1 \cup B_2 \cup B_3$.

Proof is based on general properties of Brownian paths (cf. [4]).

3. Approximation of the quasi-measure. The solution u of the Cauchy problem for the system (2) with an initial value f is expressed in the form;

$$u_j(t, x) = \int_{-\infty}^{\infty} \{f_1(y) g_{j1}^{(\alpha)}(t, x-y) + f_2(y) g_{j2}^{(\alpha)}(t, x-y)\} dy,$$

where

$$g_{jk}^{(\alpha)}(t, x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} \hat{g}_{jk}^{(\alpha)}(t, \xi) d\xi$$

with

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3) We always use a conventional notation $u_j(t, x)$ in place of $u(t, (j, x))$.

4) We write $\alpha(\cdot)$ for α. (cf. the preceding footnote).

5) The Markovian system $(z_t, \boldsymbol{P}^{(\alpha)}_{(j,x)})$ gives an affirmative answer to the question propounded by Yu. L. Daletskii and S. V. Fomin [2] whether there exists a measure which may actually be negative.

$$(\hat{g}_{jk}^{(\alpha)}(t, \xi)) = \exp\left\{t\begin{pmatrix} -\alpha_1\xi^2-\beta_1 & \beta_1 \\ \beta_2 & -\alpha_2\xi^2-\beta_2\end{pmatrix}\right\}.$$

Concerning $g_{jk}^{(\alpha)}$, we have following

Lemma 3. *If α_1 and $\alpha_2 \uparrow +\infty$ with $\alpha_1 = o(\alpha_2)$, then*

$$g_{11}^{(\alpha)}(t, x) \to g(t, x) \equiv \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x - t(a\xi^4 + b\xi^2)} d\xi,$$

$$g_{12}^{(\alpha)}(t, x) \to 0.$$

Proof. An elementary but somewhat complicated calculation shows that

$$\hat{g}_{11}^{(\alpha)}(t, \xi) = \frac{1}{\rho_+^{(\alpha)} - \rho_-^{(\alpha)}}\left[\{\rho_+^{(\alpha)} + \alpha_2\xi^2 + \beta_2\}e^{t\rho_+^{(\alpha)}} - \{\rho_-^{(\alpha)} + \alpha_2\xi^2 + \beta_2\}e^{t\rho_-^{(\alpha)}}\right],$$

$$\hat{g}_{12}^{(\alpha)}(t, \xi) = \frac{\beta_1}{\rho_+^{(\alpha)} - \rho_-^{(\alpha)}}\left[e^{t\rho_+^{(\alpha)}} - e^{t\rho_-^{(\alpha)}}\right],$$

where

$$\rho_\pm^{(\alpha)} = \frac{1}{2}\left[-\left\{(\alpha_1+\alpha_2)\xi^2 + \frac{\alpha_1\alpha_2}{a}\right\} \pm \sqrt{\left\{(\alpha_1+\alpha_2)\xi^2 + \frac{\alpha_1\alpha_2}{a}\right\}^2 - 4\alpha_1\alpha_2\left\{\xi^4 + \frac{b}{a}\xi^2\right\}}\right].$$

When α_1 and $\alpha_2 \uparrow +\infty$, then $\rho_+^{(\alpha)}(\xi) \to -a\xi^4 - b\xi^2$ and $\rho_-^{(\alpha)}(\xi) \sim -\alpha_1\alpha_2/a$. Therefore if α_1 and $\alpha_2 \uparrow +\infty$ with $\alpha_1 = o(\alpha_2)$, then

$$\frac{\beta_2}{\rho_+^{(\alpha)} - \rho_-^{(\alpha)}} \to 1$$

and the others vanish. Thus we have

$$\hat{g}_{11}^{(\alpha)}(t, \xi) \to e^{-t(a\xi^4 + b\xi^2)} \quad \text{and} \quad \hat{g}_{12}^{(\alpha)}(t, \xi) \to 0.$$

From Lemma 3 it follows

Lemma 4. *For every $A \in \mathfrak{F}$,*

$$\lim_{\alpha\to\infty} \boldsymbol{P}_{(1\ x)}^{(\alpha)}[A] = \boldsymbol{P}_x[A],$$

where $\lim\limits_{\alpha\to\infty}$ means the limit along the way such that $\alpha_1, \alpha_2 \uparrow +\infty$, $\alpha_1 = o(\alpha_2)$ and $\boldsymbol{P}_x$ is the quasi-measure defined in Introduction.

We write

$$\mathfrak{M} = \{A \in \mathfrak{B} : \lim_{\alpha\to\infty} P_{(1, x)}^{(\alpha)}[A] \text{ exists for every } x\},$$

$$\boldsymbol{P}_x[A] = \lim_{\alpha\to\infty} \boldsymbol{P}_{(1, x)}^{(\alpha)}[A] \text{ for } A \in \mathfrak{M}.$$

Then we have

Theorem. $\mathfrak{F} \subset \mathfrak{M}$. *$\boldsymbol{P}_x$ is an extension of the quasi-measure to $\mathfrak{M}$.*

$\mathfrak{M}$ is a *quasi-algebra*, i.e., it possesses the following properties;

i) $\Omega, \phi \in \mathfrak{M}$.

ii) If $A \in \mathfrak{M}$, then $\Omega \setminus A \in \mathfrak{M}$.

iii) If A and $B \in \mathfrak{M}$, then $A \cap B \in \mathfrak{M}$ is equivalent to $A \cup B \in \mathfrak{M}$.

4. Some properties of $\boldsymbol{P}_x$. From Lemma 2 follows immediately

Proposition 1. *Every $B \in \mathfrak{B}$ such that $B \subset B_1 \cup B_2 \cup B_3$ belongs to $\mathfrak{M}$ and the equality $\boldsymbol{P}_x[B] = 0$ holds, where B_1, B_2, and B_3 are those of Lemma 2.*

An analogous result is anounced by V. Yu. Krylov [6].

Let σ be the first passage time to the origin, i.e.,

$$\sigma(\omega)=\inf\{t;\, x_t(\omega)=0\}.$$

As to σ, we have

Proposition 2. *For any Borel set Γ in R^1 and for any $t\geqq 0$, $\{x_t\in\Gamma, \sigma>t\}$ and $\{x_{t\wedge\sigma}\in\Gamma\}$ belong to $\mathfrak{M}$. $\boldsymbol{P}_x[x_t\in dy, \sigma>t]$ and $\boldsymbol{P}_x[x_{t\wedge\sigma}\in dy]$ are the fundamental solutions of initial-boundary value problems for the equation* (1) *for $x>0$ with boundary conditions $u(t, 0)=0$ and $u(t, 0)=u(0, 0)$ respectively.*

Proof. It is easy to see that

$$\boldsymbol{P}^{(\alpha)}_{(1,x)}[x_t\in\Gamma, \sigma>t]=\sum_{j=1}^{2}\int_{\Gamma}\{g^{(\alpha)}_{1j}(t, x-y)-g^{(\alpha)}_{1j}(t, x+y)\}dy,$$

from which the result follows.

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