# 58. On Conditions for the Orthomodularity

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(Comm. by Kinjirô KUNUGI, M.J.A., March 12, 1966)

**1. Introduction.** The lattice of projections of a von Neumann algebra is an orthocomplemented lattice (a lattice equipped with an orthocomplementation $a\rightarrow a^{\perp}$) with a weak modularity $(M)$ introduced by Loomis [2]. Such a lattice is called an orthomodular lattice (see [3], Remark 4.1). The condition $(M)$ for the orthomodularity is equivalent to that "if $a\leqq b$ then $a, a^{\perp}, b$ satisfy some distributive relation". Piron [5] has shown that the logic of quantum mechanics forms an orthomodular lattice by the reason that "if $a\leqq b$ then the sublattice generated by $a, a^{\perp}, b, b^{\perp}$ is distributive". This condition is also equivalent to $(M)$.

On the other hand, Nakamura [4] has defined the permutability of $a, b$ by some distributive relation and proved that the condition $(M)$ is equivalent to that this permutability is symmetric. Moreover, Foulis [1] has given some other conditions like this.

The purpose of this paper is to find all the conditions of these types.

**2. D-relations.** Let $L$ be an orthocomplemented lattice where the orthocomplementation is denoted by $a\rightarrow a^{\perp}$. For $a, b, c\in L$, we write $(a, b, c)D$ in case $(a\cup b)\cap c=(a\cap c)\cup(b\cap c)$, and write $(a, b, c)D^*$ in case $(a\cap b)\cup c=(a\cup c)\cap(b\cup c)$.

**Definition.** Two elements $a, b\in L$ are said to be *commutative* if the sublattice generated by $a, a^{\perp}, b, b^{\perp}$ is distributive. We denote $aDb$ if every distributive relation for $a, a^{\perp}, b, b^{\perp}$ holds. (Obviously, if $a$ and $b$ are commutative then $aDb$.) Since $(a, b, c)D\Longleftrightarrow(b, a, c)D$ and $(a, b, c)D^*\Longleftrightarrow(a^{\perp}, b^{\perp}, c^{\perp})D$ for every $a, b, c\in L$, $aDb$ is equivalent to that the following twelve $D$-relations hold.

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|---|---|---|
| $D_1:(a, a^{\perp}, b)D$ | $D_{13}:(b^{\perp}, a^{\perp}, a)D$ | $D_{14}:(b^{\perp}, a, a^{\perp})D$ |
| $D_2:(a, a^{\perp}, b^{\perp})D$ | $D_{23}:(b, a^{\perp}, a)D$ | $D_{24}:(b, a, a^{\perp})D$ |
| $D_3:(b, b^{\perp}, a)D$ | $D_{31}:(a^{\perp}, b^{\perp}, b)D$ | $D_{32}:(a^{\perp}, b, b^{\perp})D$ |
| $D_4:(b, b^{\perp}, a^{\perp})D$ | $D_{41}:(a, b^{\perp}, b)D$ | $D_{42}:(a, b, b^{\perp})D$ |

**Lemma 1.** *$D_i$ implies $D_{ij}$* $(i=1, 2$ and $j=3, 4$; $j=3, 4$ and $j=1, 2)$.

**Proof.** $D_1$ means $b=(a\cap b)\cup(a^{\perp}\cap b)$. From this, we have $b\cup a^{\perp}=(a\cap b)\cup a^{\perp}$, $b\cup a=(a^{\perp}\cap b)\cup a$, and hence $b^{\perp}\cap a=(a^{\perp}\cup b^{\perp})\cap a$, $b^{\perp}\cap a^{\perp}=(a\cup b^{\perp})\cap a^{\perp}$ by the orthocomplementation. Therefore, $D_{13}$ and $D_{14}$ hold. The other implications can be proved similarly.

**Lemma 2.** (i) *If* $a \leqq b$, *then* $D_1$ (*resp.* $D_4$) *is equivalent to* $D_{14}$ (*resp.* $D_{41}$) *and the other eight D-relations hold.*

(ii) *If* $b \leqq a$, *then* $D_2$ (*resp.* $D_3$) *is equivalent to* $D_{23}$ (*resp.* $D_{32}$) *and the other eight D-relations hold.*

(iii) *If* $a \leqq b^{\perp}$, *then* $D_2$ (*resp.* $D_4$) *is equivalent to* $D_{24}$ (*resp.* $D_{42}$) *and the other eight D-relations hold.*

(iv) *If* $b^{\perp} \leqq a$, *then* $D_1$ (*resp.* $D_3$) *is equivalent to* $D_{13}$ (*resp.* $D_{31}$) *and the other eight D-relations hold.*

**Proof.** (i) If $a \leqq b$, then $b^{\perp} \leqq a^{\perp}$, and $a \cap b^{\perp} = 0$. Hence, we have $(a \cap b^{\perp}) \cup (a^{\perp} \cap b^{\perp}) = b^{\perp} = (a \cup a^{\perp}) \cap b^{\perp}$ and $(b \cap a) \cup (b^{\perp} \cap a) = a = (b \cup b^{\perp}) \cap a$, that is, $D_2$ and $D_3$ hold. It follows from Lemma 1 that $D_{23}$, $D_{24}$, $D_{31}$, $D_{32}$ hold. Moreover, $D_{13}$ and $D_{42}$ hold since $(b^{\perp} \cap a) \cup (a^{\perp} \cap a) = 0 = (b^{\perp} \cup a^{\perp}) \cap a$ and $(a \cap b^{\perp}) \cup (b \cap b^{\perp}) = 0 = (a \cup b) \cap b^{\perp}$. Next, since $D_1$ and $D_{14}$ mean the relations $b = a \cup (a^{\perp} \cap b)$ and $(b^{\perp} \cup a) \cap a^{\perp} = b^{\perp}$ respectively, they are equivalent by the orthocomplementation. Similarly, $D_4$ and $D_{41}$ are equivalent. (ii) is implied from (i) by the exchange $a \leftrightarrow b$. (iii) and (iv) are implied from (i) and (ii) by the exchange $b \leftrightarrow b^{\perp}$.

**3. Conditions for the orthomodularity. Definition.** A pair $(a, b)$ of elements of a lattice is called a *modular pair* and write $(a, b)M$ if $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$. An orthocomplemented lattice $L$ is called *orthomodular* if $(a, a^{\perp})M$ holds for every $a \in L$, or equivalently, if $a \perp b$ $(a \leqq b^{\perp})$ implies $(a, b)M$ (see [3], Theorem 4.1 and Remark 4.1).

**Theorem 1.** *Let L be an orthocomplemented lattice. The following statements are equivalent.*

($\alpha$) *L is orthomodular.*

($\beta_1$) (*resp.* ($\beta_1'$), ($\beta_1''$), ($\beta_1'''$)) *If* $a \leqq b$, *then* $D_1$ (*resp.* $D_4$, $D_{14}$, $D_{41}$) *holds.*

($\beta_2$) (*resp.* ($\beta_2'$), ($\beta_2''$), ($\beta_2'''$)) *If* $b \leqq a$, *then* $D_2$ (*resp.* $D_3$, $D_{23}$, $D_{32}$) *holds.*

($\beta_3$) (*resp.* ($\beta_3'$), ($\beta_3''$), ($\beta_3'''$)) *If* $a \leqq b^{\perp}$, *then* $D_2$ (*resp.* $D_4$, $D_{24}$, $D_{42}$) *holds.*

($\beta_4$) (*resp.* ($\beta_4'$), ($\beta_4''$), ($\beta_4'''$)) *If* $b^{\perp} \leqq a$, *then* $D_1$ (*resp.* $D_3$, $D_{13}$, $D_{31}$) *holds.*

($\gamma$) *If* $a \leqq b$, *then* $aDb$.

($\delta$) *If* $a \leqq b$, *then a and b are commutative.*

**Proof.** The implications $(\delta) \Rightarrow (\gamma) \Rightarrow (\beta_i^{(\nu)})$ $(i = 1, 2, 3, 4;\ \nu = 0, 1, 2, 3)$ are trivial. $(\beta_1) \Rightarrow (\gamma)$. Assume that $a \leqq b$ implies $D_1$: $(a, a^{\perp}, b)D$. Then, since $a \leqq b \Longleftrightarrow b^{\perp} \leqq a^{\perp}$, $a \leqq b$ implies $D_4$: $(b^{\perp}, b, a^{\perp})D$. Hence, it follows from Lemma 2 (i) that $a \leqq b$ implies all $D$-relations. The other implications $(\beta_i^{(\nu)}) \Rightarrow (\gamma)$ can be proved similarly. $(\gamma) \Rightarrow (\delta)$. If $a \leqq b$ and $(\gamma)$ holds, then we have $a \cup (a^{\perp} \cap b) = b$ and $b \cup (b^{\perp} \cap a) = a$.

Then, the eight elements $\{0, a, a^{\perp}\cap b, b^{\perp}, b, a\cup b^{\perp}, a^{\perp}, 1\}$ form a distributive sublattice, and hence $a$ and $b$ are commutative. $(\alpha)\Longleftrightarrow(\beta)$. $(a, a^{\perp})M$ means that $b\leqq a$ implies $b=(b\cup a^{\perp})\cap a$, that is, $b\leqq a$ implies $D_{23}$. Hence $(\alpha)\Longleftrightarrow(\beta_2'')$. This completes the proof.

**Remark 1.** The condition $(M)$ in Loomis [2] means that $a\leqq b$ implies $(a^{\perp}, b, a)D^*$, that is, $a\leqq b$ implies $D_{14}$. The condition $(M_2)$ means that $a\leqq b^{\perp}$ implies $D_{42}$. The condition "faiblement modulaire" in Piron [5] means that $a\leqq b$ implies $D_{41}$.

**Definition.** In an orthocomplemented lattice $L$, we shall call the eight implications "$D_i\Rightarrow D_{ij}$" ($i=1, 2, j=3, 4; i=3, 4, j=1, 2$) *D-implications of type I*, the eight implications "$D_i\Rightarrow D_{ji}$" *D-implications of type II*, the eight implications "$D_{ij}\Rightarrow D_{ji}$" *D-implications of type III* and the other 108 implications *D-implications of type IV*. (The total number of $D$-implications is ${}_{12}P_2=132$.)

It follows from Lemma 1 that $D$-implications of type I always hold, and it is easy to show by the exchanges $(a, b)\leftrightarrow(b, a)$, $(a, b)\leftrightarrow(a, b^{\perp})$, $(a, b)\leftrightarrow(a^{\perp}, b)$ that $D$-implications of type II are mutually equivalent and so are $D$-implications of type III.

**Theorem 2.** *Let L be an orthocomplemented lattice. The following statements are equivalent.*

$(\alpha)$ *L is orthomodular.*

$(\beta)$ *One of the D-implications of type IV holds.*

$(\gamma)$ *All the D-implications hold, that is, all the D-relations are mutually equivalent.*

**Proof.** $(\gamma)\Rightarrow(\beta)$ is trivial. We shall prove $(\beta)\Rightarrow(\alpha)$. For example, let "$D_1\Rightarrow D_2$" hold. If $b\leqq a$, then $D_1$ holds by Lemma 2 (ii) and then $D_2$ holds. It follows from Theorem 1 ($(\beta_2)\Rightarrow(\alpha)$) that $L$ is orthomodular. If we assume one of the other $D$-implication of type IV, then similarly we can prove that $L$ is orthomodular by Lemma 2 and Theorem 1. To prove $(\alpha)\Rightarrow(\gamma)$, we shall show that if $L$ is orthomodular then $D_{ij}\Rightarrow D_j$, for example $D_{13}\Rightarrow D_3$. It follows from $(a\cap b, a^{\perp}\cup b^{\perp})M$ that $[a^{\perp}\cup(a\cap b)]\cap(a^{\perp}\cup b^{\perp})=a^{\perp}$, which implies $a=(a\cap b)\cup[a\cap(a^{\perp}\cup b^{\perp})]$. It follows from $D_{13}$ that $(b^{\perp}\cup a^{\perp})\cap a=b^{\perp}\cap a$. Hence $a=(a\cap b)\cup(a\cap b^{\perp})$ which means $D_3$ holds. For every $i$, $j$, we have $D_{ij}\Rightarrow D_j$ by the same way. Now, since $D_i\Rightarrow D_{ij}$ by Lemma 1, we have the following cyclic implications: $D_i\Rightarrow D_{ij}\Rightarrow D_j\Rightarrow D_{ji}\Rightarrow D_i$. Hence all the $D$-relations are equivalent. This completes the proof.

**Remark 2.** The condition "symmetric" in Nakamura [4] is "$D_3\Rightarrow D_1$". The conditions given by Foulis [1] are "$D_1\Rightarrow D_{23}$" and "$D_{41}\Rightarrow D_{23}$".

Corollary. *Let a, b be elements of an orthomodular lattice L. The following statements are equivalent.*

($\alpha$) *a and b are commutative.*

($\beta$) *aDb.*

($\gamma$) *One of the twelve D-relations holds.*

**Proof.** The implications $(\alpha)\Rightarrow(\beta)\Rightarrow(\gamma)$ are trivial. $(\gamma)\Rightarrow(\beta)$ is an immediate consequence of the theorem. $(\beta)\Rightarrow(\alpha)$ is a consequence of [1], Lemma 3 and Theorem 5.

**Theorem 3.** *For two elements a, b of an orthocomplemented lattice L, we write* $a\leftrightarrow b$ *in case* $a\cup(b\cap a^{\perp})=b\cup(a\cap b^{\perp})$ (see Piron [5]). *The following statements are equivalent.*

($\alpha$) *L is orthomodular.*

($\beta_1$) *If* $a\leqq b$ *then* $a\leftrightarrow b$.

($\beta_2$) *If* $a\leqq b$ *then* $a^{\perp}\leftrightarrow b^{\perp}$.

($\gamma$) $a\leftrightarrow b$ *implies* $a\leftrightarrow b^{\perp}$.

($\delta$) $a\leftrightarrow b$ *implies aDb.*

**Proof.** $a\leftrightarrow b$ is equivalent to both of the two equations $a\cup(b\cap a^{\perp})=a\cup b$ and $b\cup(a\cap b^{\perp})=a\cup b$, that is, $D_{14}$ and $D_{32}$. Hence, ($\beta_1$) implies ($\beta_1''$) of Theorem 1 and is implied from ($\gamma$) of Theorem 1. Therefore, $(\beta_1)\Longleftrightarrow(\alpha)$. $(\beta_1)\Longleftrightarrow(\beta_2)$ is obvious. $(\alpha)\Rightarrow(\delta)$ follows from Theorem 2, and $(\delta)\Rightarrow(\gamma)$ is trivial. Finally, we assume ($\gamma$). If $a\leqq b^{\perp}$, then $a\leftrightarrow b$ holds by Lemma 2 (iii), and then we have $a\leftrightarrow b^{\perp}$, which implies $D_{24}$. Hence, $L$ is orthomodular by Theorem 1. This completes the proof. (The main part of this theorem has proved by Piron.)

**Remark 3.** (i) The implications "$a\leqq b\Rightarrow a\leftrightarrow b^{\perp}$" and "$a\leqq b\Rightarrow a^{\perp}\leftrightarrow b$" always hold.

(ii) "$a\leftrightarrow b\Rightarrow a^{\perp}\leftrightarrow b^{\perp}$" is not equivalent to the orthomodularity, since it is implied from $D$-implications of type III (cf. Supplement).

**Corollary.** *Let a, b be elements of an orthomodular lattice.* $a\leftrightarrow b$ *if and only if a and b are commutative.*

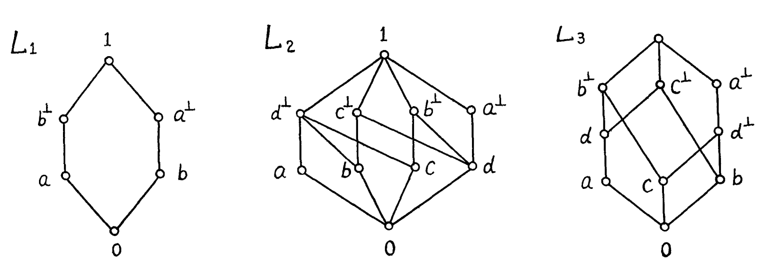
**4. Supplement.** We consider the following four statements.

($\alpha$) $L$ is orthomodular.

($\beta$) $L$ is orthocomplemented and the $D$-implications of type III hold.

($\gamma$) $L$ is orthocomplemented and the $D$-implications of type II hold.

($\delta$) $L$ is orthocomplemented.



Then we have implications $(\alpha)\Rightarrow(\beta)\Rightarrow(\gamma)\Rightarrow(\delta)$. The preceding figures give examples such that $(\alpha)\not\Leftarrow(\beta)\not\Leftarrow(\gamma)\not\Leftarrow(\delta)$.

In the lattice $L_1$, for any two elements $x$, $y$, we have $x\leqq y$ or $y\leqq x$ or $x\leqq y^{\perp}$ or $y^{\perp}\leqq x$. Hence, $L_1$ satisfies $(\beta)$ by Lemma 2, but is not orthomodular. In the lattice $L_2$, for the elements $a$ and $b$, $D_{24}$ holds but $D_{42}$ does not. Hence, $L_2$ does not satisfy $(\beta)$. For $a$ and $b$, $D_1$, $D_2$, $D_3$, and $D_4$ do not hold. Hence, it is easy to verify that $L_2$ satisfies $(\gamma)$. In the lattice $L_3$, for $a$ and $b$, $D_2$ holds but $D_{42}$ does not. Hence, $L_3$ does not satisfy $(\gamma)$, but is orthocomplemented.

## References

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