54. *Connection of Topological Fibre Bundles. II*

By Akira ASADA

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In his note [2], the author defined the connection forms for an arbitrary topological fibre bundle ξ to be an element in $C^1(X_G, G)$ such that $s(\alpha a, \beta b)=a^{-1}s(\alpha, \beta)b$, where X_G is the total space of the principal bundle of ξ, and a, b are elements of G, the structure group of ξ. There, first we define the obstruction class for the existence of (topological) connection forms (of. [3]). Next we consider a relation between the topological curvature forms of complex vector bundles and their complex Chern classes (cf. [5]). We use the same notations as [2] in this note. For example, we denote

$C^1(X_G, G)_G=\{s \mid s\in C^1(X_G, G), s(\alpha a, \beta b)=a^{-1}s(\alpha, \beta)b, a, b\in G\}$,

$T^1(X_G, G)=\{s \mid s\in C^1(X_G, G), s(\alpha a, \beta b)=a^{-1}s(\alpha, \beta)a, a, b\in G\}$,

$T^2(X_G, G)=\{s \mid s\in C^2(X_G, G), s(\alpha a, \beta b, \gamma c)=b^{-1}s(\alpha, \beta, \gamma)b, a, b, c\in G\}$.

1. *Obstruction class for the existence of topological connection forms.* We denote by X_G the total space of the principal bundle associated to a topological G-bundle ξ over X, π the projection of X_G to X. If U is a coordinate neighborhood of ξ then, by lemma 4 of [2], $C^1(\pi^{-1}(U), G)_G$ is not an empty set and we obtain by the corollary of theorem 2 in [2]

$$C^1(\pi^{-1}(U), G)_G= T^1(\pi^{-1}(U), G)s, \tag{1}$$

where s is a connection form of $\xi \mid U$.

On X_G, we set

$\mathcal{S}^1$: the sheaf of germs of elements of $C^1(\pi^{-1}(U), G)_G$,

$\mathcal{T}^i$: the sheaf of germs of elements of $T^i(\pi^{-1}(U), G)$, $i=1, 2$.

If we regard $\mathcal{S}^1$ and $\mathcal{T}^i$ to be sheaves on X, then we denote them by $\mathcal{S}^1_\xi, \mathcal{T}^i_\xi$ and call that $\mathcal{S}^1_\xi$ is the connection sheaf and $\delta_1\mathcal{S}^1_\xi$ is the curvature sheaf of ξ.

Since $T^i(\pi^{-1}(U), G)$ are groups, $\mathcal{T}^i$ are sheaves of groups for $i=1, 2$, but $\mathcal{S}^1$ is only a sheaf of sets. But by (1), if s_U belongs to $H^0(\pi^{-1}(U), \mathcal{S}^1)$, then $s_U s_V^{-1}$ belongs to $H^0(\pi^{-1}(U\cap V), \mathcal{T}^1)$ and we get

Lemma 1. *The class of* $\{s_U s_V^{-1}\}$ *in* $H^1(X_G, \mathcal{T}^1)$ *does not depend on the choice of* $\{s_U\}$.

Definition. The class of $\{s_U s_V^{-1}\}$ in $H^1(X_G, \mathcal{T}^1)$ is called the obstruction class for the existence of (topological) connection of ξ and denoted by $o(\xi)$.

Theorem 1. *ξ has a connection form if and only if $o(\xi)$ is equal to 1 in $H^1(X_G, \mathcal{T}^1)$.*

Note 1. *If* ξ is an analytic bundle and restrict $C^1(\pi^{-1}(U), G)_G$ and $T^1(\pi^{-1}(U), G)$ to the sets of holomorphic maps, then ξ has a holomorphic connection if and only if $o(\xi)$ is equal to 1 (cf. [3]).

Note 2. We denote by $[\mathcal{S}^1]$ the sheaf of groups generated by $\mathcal{S}^1$, then $\mathcal{T}^1$ is a subsheaf of $[\mathcal{S}^1]$ and ξ has a connection form if and only if the sequence

$$0 \longrightarrow \mathcal{T}^1 \longrightarrow [\mathcal{S}^1] \longrightarrow [\mathcal{S}^1]/\mathcal{T}^1 \longrightarrow 0,$$

splits.

2. *Calculation of* $H^1(X_G, \mathcal{T}^1)$ *in some special cases.* We use the following notation.

G^r: the sheaf of germs of elements of $C^r(U, G)$, $r \geq 0$.

Under this notation, we get if G is an abelian group,

$$H^1(X_G, \mathcal{T}^1) \simeq H^1(X, G^1), \tag{2}$$

because $a^{-1}ba = b$ in this case.

To calculate $H^1(X, G^1)$, we assume G is a connected, locally connected and locally compact abelian group. Then by [8], theorem 42, we have

$$G \simeq R^\mu \times T^\nu,$$

where R^μ is the μ-direct product of R^1, the additive group of real numbers, T^ν is the ν-direct product of $T^1 = R^1/Z$, and we obtain

Lemma 2. *Under the above assumption, we get*

$$G^r \simeq (R^{\mu+\nu})^r, \qquad r \geq 1. \tag{3}$$

Corollary 1. *If G is a connected, locally connected and locally compact abelian group, and if X is a normal paracompact topological space then $H^1(X_G, \mathcal{T}^1)$ vanishes.*

Corollary 2. *If X and G are the same as corollary 1, then a topological G-bundle over X always has a topological connection form.*

Note 1. Since T^1 can not be imbedded in R^1, the corollary 2 is not a special case of [1], theorem 1.

Note 2. The isomorphism (3) is true even if G^r and $(R^{\mu+\nu})^r$ are both sheaves of holomorphic sections. But (3) is false if $r=0$ or if we use $\tilde{C}^r(U, G)$ instead of $C^r(U, G)$ in the definition of G^r.

To extend this result for a non abelian group G, we assume H is a (closed) normal subgroup of G and set $G/H = A$. We denote

$\mathcal{T}^i_H$: the subsheaf of $\mathcal{T}^i$ consisting of those germs that their values belong to H, $i=1, 2$.

Then, setting $\mathcal{T}^1/\mathcal{T}^1_H \simeq {}_A\mathcal{T}^1$, we get

$$H^0(\pi^{-1}(U), {}_A\mathcal{T}^1) = \{s \mid s(\alpha, \beta) \in A,\ s(\alpha a, \beta b) = \bar{a}^{-1}s(\alpha, \beta)\bar{a},\ a, b \in G\},$$

where $\bar{a}$ means the class of a mod. H. Then as we know the sequence

$$H^1(X_G, \mathcal{T}^1_H) \longrightarrow H^1(X_G, \mathcal{T}^1) \longrightarrow H^1(X_G, {}_A\mathcal{T}^1)$$

is exact, we obtain by the corollary 1 of lemma 2, the following

Theorem 2. *If G is a connected, locally connected and locally compact solvable group, and if X is a normal paracompact topological space, then $H^1(X_G, \mathfrak{T}^1)$ vanishes.*

Corollary. *If G is a connected, locally connected and locally compact solvable group, and if X is a normal paracompact topological space, then a topological G-bundle over X always has a connection form.*

3. *Curvature form of an abelian bundle.* In this n^0, we assume that G is a connected, locally connected and locally compact abelian group and that X is a normal paracompact topological space.

By (5) of [2], if s is a connection form of an arbitrary fibre bundle η (the structure group of η need not be an abelian group), then we get

$$(4)\quad \delta_1(ts)(\alpha, \beta, \gamma) = (\delta_1 s)(\alpha, \beta, \gamma)t(\beta\delta_1 s(\alpha, \beta, \gamma), \gamma)t(\beta, \gamma)^{-1}(\delta_1 t)(\alpha s(\alpha, \beta), \beta, \gamma),$$
$$s \in C^1(Y_G, G)_G,\ t \in T^1(Y_G, G),$$

where Y is the base space of η (Y need not be a normal paracompact space). Therefore if s is a connection form of ξ then we can regard its curvature form $\delta_1 s$ to be an element of $C^2(X, G)$ because G is an abelian group. $\delta_1 s$ regarded as an element of $C^2(X, G)$ is denoted by $(\delta_1 s)^\natural$. Moreover, since δ is natural, $(\delta_1 s)^\natural$ belongs to $Z^2(X, G)$ and by the corollary of theorem 2 of [2], we obtain

Lemma 3. *The class of $(\delta_1 s)^\natural$ in $Z^2(X, G)/B^2(X, G)$ does not depend on the choice of s.*

Here $B^2(X, G) = \delta_1 C^1(X, G)$.

On the other hand, by this lemma and theorem 3 of [2] (or corollary 2 of theorem 2 of [1]), we obtain

Lemma 4. *ξ is induced from a representation of $\pi_1(X)$ into G if and only if ξ has a connection form s such that $(\delta_1 s)^\natural$ belongs to $B^2(X, G)$.*

Since we get

$$(5)\quad Z^2(X, G)/B^2(X, G) \simeq H^2(X, R^{\mu+\nu}),$$

by lemma 2 and $n^\circ 1$ of [1], we obtain by lemma 3 and 4

Theorem 3. *The following sequence is exact.*

$$(6)\quad H^1(X, G) \xrightarrow{\ i\ } H^1(X, G^0) \xrightarrow{\ \chi\ } H^2(X, R^{\mu+\nu}).$$

Here i is the map induced from the inclusion $i: G \to G^0$, where G means the sheaf of germs of constant G-valued maps, χ is the map defined by

$$(7)\quad \chi(\xi) = \text{the class of } (\delta_1 s)^\natural \text{ in } H^2(X, R^{\mu+\nu}),$$

and $H^1(X, G^0)$ is the group of all topological G-bundles on X (cf. [6]).

Example. We assume that $G = C^*$, the multiplicative group of all complex numbers except 0, and the isomorphism (3) is induced from the exact sequence of sheaves

$$0 \longrightarrow Z \xrightarrow{i} C^0 \xrightarrow{j} C^{*0} \longrightarrow 0, \qquad j(c) = \exp.(c)/2\pi\sqrt{-1},$$

that is the diagrams

$$\begin{array}{ccccc} C^0 & \xrightarrow{p_i^*} & \tilde{C}^1 & \xleftarrow{i} & C^1 \\ \downarrow j & & \downarrow j' & & \downarrow j'' \\ C^{*0} & \xrightarrow{p_i^*} & \tilde{C}^{*1} & \xleftarrow{i} & C^{*1}, \end{array} \qquad p_i(x_1, x_2) = x_i,\ i = 1, 2,$$

are both commutative, then the following diagram is commutative.

$$\begin{array}{ccc} & & H^2(X, Z) \\ & \overset{\delta}{\nearrow} & \downarrow i' \\ H^1(X, C^{*0}) & \xrightarrow{\chi} & H^2(X, C), \end{array}$$

where δ is the coboundary homomorphism induced from the above exact sequence. Therefore if ξ is a complex line bundle, then $\chi(\xi)$ is its first complex Chern class because $\delta(\xi)$ is the first integral Chern class of ξ (cf. [6]).

4. *Topological curvature forms and characteristic classes.* First we assume that $G = \Delta(n, C)$, where $\Delta(n, C)$ is the subgroup of $GL(n, C)$ consisting of those matrices with the form

$$M = \begin{pmatrix} * & & * \\ 0 & \ddots & \\ \vdots & \ddots & \ddots \\ 0 & \cdots\ 0 & * \end{pmatrix}.$$

We assume that X is a normal paracompact topological space. Then a $\Delta(n, C)$-bundle ξ over X always has a connection form by the corollary of theorem 2. Moreover, the curvature form $\delta_1 s$ of ξ defines n-elements $c_1, \cdots, c_n$ of $H^2(X, C)$.

If f belongs to $C^2(Y, GL(n, C))$, where Y is a topological space, then denoting the proper values of f at (x_0, x_1, x_2) by $\lambda_1(x_0, x_1, x_2), \cdots, \lambda_n(x_0, x_1, x_2)$, we set

$$\begin{aligned} \hat{\chi}_t(f) = 1 &+ (t/2\pi\sqrt{-1}) \sum_{i=1}^{n} \lambda_i(x_0, x_1, x_2) \\ &+ (t/2\pi\sqrt{-1})^2 \sum_{i<j} \lambda_i(x_0, x_1, x_2)\lambda_j(x_2, x_3, x_4) + \cdots \\ &+ (t/2\pi\sqrt{-1})^n \lambda_1(x_0, x_1, x_2)\lambda_2(x_2, x_3, x_4) \cdots \lambda_n(x_{2n-2}, x_{2n-1}, x_{2n}). \end{aligned}$$

By definition, the class of the coefficients of $\hat{\chi}_t(f)$ in $\sum_{i\geq 0} C^i(Y, C)/A$ does not depend on the order of $\lambda_1, \cdots, \lambda_n$. Here A means the alternating ideal of $\sum_{i\geq 0} C^i(Y, C)$. We denote by $\chi_t(f)$ the polynomial in $\sum_{i\geq 0} (C^{2i}(Y, C)/A)t^i$ induced from $\hat{\chi}_t(f)$. Since the proper values of a matrix M are determined by the conjugate class of M, we can regard $\chi_t(\delta_1 s)$ to be a polynomial in $\sum_{i\geq 0} (C^{2i}(X, C)/A)t^i$ if s is a connection form of a $GL(n, C)$-bundle ξ over X by (5) of [2].

Moreover, we obtain by (4), if ξ is a $\Delta(n, C)$-bundle, s is a connection form in $\Delta(n, C)$, then each coefficient of $\chi_t(\delta_1 s)$ is a cocycle and we get

$$(8)\qquad \chi_t(\delta_1 s) \bmod. \sum_{i\geq 0} (B^{2i}(X, C)/A)t^i = \prod_{i=1}^{n} (1+c_i t).$$

In general, setting $F(n)=GL(n, C)/\Delta(n, C)$ and denote the projection from the associated $F(n)$-bundle of ξ to X by $\pi_{F(n)}$, we obtain by [7],

Theorem 4′ (cf. [5]). *If s is a connection of a $GL(n, C)$-bundle ξ over X such that $\pi_{F(n)}{}^*(s)$ is equivalent to a $\Delta(n, C)$-valued connection of $\pi_{F(n)}{}^*(\xi)$, then the coefficients $\chi_t(\delta_1 s)$ are belong to $\sum_{i\geq 0} Z^{2i}(X, C)/A$ and their classes* mod. $\sum_{i\geq 0} B^{2i}(X, C)/A$ *do not depend on the choice of s if X is compact.*

5. *The group $K_\theta(X)$.* To show the coefficients of $\chi_t(\delta_1 s)$ give the complex Chern classes of ξ, a $GL(n, C)$-bundle over X, at least for compact X, we define the group $K_\theta(X)=K_{GL(n,C),\theta}(X)$ to be the Grothendieck group generated by (the equivalence class of) $\delta_1(\mathcal{S}^1_\xi)$ for all $GL(n, C)$-bundles ξ over X (cf. [4], 4). Then we obtain

Lemma 5. *If ξ_1, ξ_2 are $GL(n, C)$-bundles on X, $\delta_1(\mathcal{S}^1_{\xi_1})$ and $\delta_1(\mathcal{S}^1_{\xi_2})$ are their curvature sheaves, then $\delta_1(\mathcal{S}^1_{\xi_1}\oplus\mathcal{S}^1_{\xi_2})$ is the curvature sheaf of $\xi_1\oplus\xi_2$.*

Note. More precisely, if $\{s^1_U\}$ is a connection form of ξ_1 and $\{s^2_U\}$ is a connection form of ξ_2 then $\{s^1_U\oplus s^2_U\}$ is a connection form of $\xi_1\oplus\xi_2$.

By this lemma and theorem 3 of [2] or corollary 2 of theorem 2 of [1], we obtain

Lemma 6. *The following sequence is exact.*

$$(9)\qquad 0 \longrightarrow K^0_h(X) \xrightarrow{\ i\ } K^0_C(X) \xrightarrow{\ j\ } K_\theta(X) \longrightarrow 0,$$

where $K^0_h(X)=K^0_{h,GL(n,C)}(X)$ is the subgroup of $K^0_C(X)$ generated by those bundles that are induced from representations of $\pi_1(X)$ into $GL(n, C)$, i is the inclusion map and j is the map defined by

$$(10)\qquad j([\xi])=[\delta_1(\mathcal{S}^1_\xi)],$$

if $[\xi]$ is the class of a vector bundle ξ.

Note. Similar exact sequence is true for $K^0_R(X)$ (cf. [1], theorem 1 and 2). Moreover, although we do not know whether there exists or not a connection form for a topological microbundle $\mathfrak{X}$ over X, we can define the curvature sheaf of $\mathfrak{X}$ to be a non-empty set, if X is a normal paracompact topological space. Because we can define a sheaf of groups $\mathcal{F}$ and if X is a normal paracompact topological space then $\mathfrak{X}$ is expressed as an element of $H^1(X, \mathcal{F})$. (The converse of this fact is also true). Therefore we can define $K_{\text{top.}\theta}(X)$ to be the Grothendieck group generated by the curvature sheaves of topological microbundles over X if X is a normal paracompact topological space. Then we get the following exact sequence.

$$K^0_{\text{top.}}(X) \xrightarrow{j'} K_{\text{top.}\Theta}(X) \longrightarrow 0, \tag{9}$$
$$\ker. j' \supset K^0_{h,H_*(n)}(X),$$

where $K^0_{h,H_*(n)}(X)$ is the subgroup of $K^0_{\text{top.}}(X)$ generated by those microbundles that are induced from representations of $\pi_1(X)$ into $H_*(n)$, the group of germs at the origin of those homeomorphism f from R^n into R^n that $f(0)=0$.

Since we know

$$\chi_t(\delta_1 s \oplus \delta_1 s') = \chi_t(\delta_1 s)\chi_t(\delta_1 s'), \tag{11}$$

we have by theorem 4′ and example of $n°3$ (cf. [6]),

Theorem 4. *If X is compact and $\pi_{F(n)}{}^*(s)$ is equivalent to a $\Delta(n, C)$-valued connection of $\pi_{F(n)}{}^*(\xi)$, then the complex cohomology classes of the coefficients of $\chi_t(\delta_1 s)$ give the complex Chern classes of ξ.*

Note. If ξ is a complex analytic vector bundle, $\mathcal{S}^1$ and $\mathcal{T}^1$ are the sheaves of germs of holomorphic maps, then using $\chi_t(o(\xi))$, we can define the characteristic classes of ξ (cf. [3]).

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