47. On the Existence of Competitive Equilibrium

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The purpose of this note is to show the existence of competitive equilibrium for an economy, where the excess demand function is supposed to be a point-to-set mapping, without the aid of fixed point theorems.[1]

First, the economic model in question will be specified with the help of the following notations and terminology, where all commodities are labeled $i=1, 2, \cdots, n$;

X: the commodity space (mathematically, an n-dimensional Euclidean space R^n);[2]

P: the set of price vectors (mathematically, a R^n_+ with the origin 0 deleted);

$E(p)$: the excess demand function (mathematically, a point-to-set mapping from P into X).

$p^* \in P$ will be called an *equilibrium price vector*, if there exists $x^* \in E(p^*)$ such that $0 \geqq x^*$. Our main concern is with the existence of such equilibrium price vectors. To this end, the following assumptions may be imposed on $E(p)$:

(C) $E(p)$ is continuous on P, i.e., both upper semi-continuous and lower semi-continuous on P. Furthermore the set $E(p)$ is compact for all $p \in S$;

(H) $E(p)$ is positive homogeneous of degree zero, i.e.,

$$E(\lambda p)=E(p) \qquad \text{for all } \lambda>0 \text{ and } p \in P;$$

(W) The generalized Walras law holds, i.e.,

$$(p, x)^{3)} \leqq 0 \qquad \text{for all } p \in P \text{ and } x \in E(p);$$

(S) Weak gross substitutability prevails, i.e., $p \geqq q$ and $p_i = q_i$ imply that $x_i \geqq y_i$ holds for any $x \in E(p)$ and any $y \in E(q)$ ($i = 1, 2, \cdots, n$).

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1) Similar developments are found in the following papers. H. Nikaido: Generalized gross substitutability and extremization, in Advances in Game Theory, Princeton U. P., 55-68 (1964). K. Kuga: Weak gross substitutability and the existence of competitive equilibrium, in Econometrica, **33**, 593-599 (1965).

2) The element of R^n may be considered as the row vector. $0=(0, 0, \cdots, 0)$. $e=(1, 1, \cdots, 1)$. For $x=(x_1, x_2, \cdots, x_n)$ and $y=(y_1, y_2, \cdots, y_n)$ $x \geqq y$ means $x_i \geqq y_i$ for $i=1, 2, \cdots, n$. R^n_+ denotes the set $\{p | p \in R^n, p \geqq 0\}$. S denotes the set $\left\{p \mid p \in P, \sum_{i=1}^n p_i = 1\right\}$.

3) $(p, x) = \sum_{i=1}^n p_i x_i$, where $p=(p_1, p_2, \cdots, p_n)$ and $x=(x_1, x_2, \cdots, x_n)$.

Now we have all concepts needed to state

Theorem. *An economy with $E(p)$ satisfying (C), (H), (W), and (S) has an equilibrium price vector.*

Proof. For $\varepsilon>0$, let M^ε be the set of all $\mu \geqq 0$ such that

$$\max[0, x]^{4)} + \varepsilon e \geqq \mu p \text{ for some } p \in S \text{ and some } x \in E(p).$$

Clearly $\varepsilon \in M^\varepsilon$. Since the set $E(S)=\bigcup_{p\in S} E(p)$ is compact, there exists a positive number α such that $\alpha e \geqq \max[0, x]+\varepsilon e$ for all $p \in S$ and all $x \in E(p)$. Hence M^ε is bounded.

Next consider any sequence $\{\mu^\nu\}$ such that $\lim_{\nu\to\infty} \mu^\nu=\bar{\mu}$ and $\mu^\nu \in M^\varepsilon$ for all ν. Then there exist sequences $\{p^\nu\}$ and $\{x^\nu\}$ such that $p^\nu \in S$, $x^\nu \in E(p^\nu)$ and $\max[0, x^\nu]+\varepsilon e \geqq \mu^\nu p^\nu$. Since S and $E(S)$ are compact, we may without loss of generality that $\lim_{\nu\to\infty} p^\nu=\bar{p} \in S$ and $\lim_{\nu\to\infty} x^\nu= \bar{x} \in E(S)$. Then, by the upper semi-continuity of $E(p)$, we have $\bar{x} \in E(\bar{p})$. Moreover, $\max[0, \bar{x}]+\varepsilon e \geqq \bar{\mu}\bar{p}$ and $\bar{\mu} \geqq 0$. Therefore $\bar{\mu} \in M^\varepsilon$. Thus it has been shown that M^ε is closed.

Putting $\lambda^\varepsilon=\sup M^\varepsilon$, it follows from the closedness of M^ε that $\lambda^\varepsilon \in M^\varepsilon$. Hence there exist $p^\varepsilon \in S$ and $x^\varepsilon \in E(p^\varepsilon)$ such that

$$(1) \qquad \max[0, x^\varepsilon]+\varepsilon e \geqq \lambda^\varepsilon p^\varepsilon.$$

It can be shown that equality holds in (1). Assume the contrary and suppose, after a suitable renumbering, that the following system of inequalities holds for some $m(0<m<n)$:

$$(2) \qquad \max[0, x_i^\varepsilon]+\varepsilon > \lambda^\varepsilon p_i^\varepsilon \text{ for } i=1, 2, \cdots, m,$$

and

$$(3) \qquad \max[0, x_i^\varepsilon]+\varepsilon = \lambda^\varepsilon p_i^\varepsilon \text{ for } i=m+1, \cdots, n.$$

Let $p^\nu=(p_1^\varepsilon+1/\nu, p_2^\varepsilon, p_3^\varepsilon, \cdots, p_n^\varepsilon)$. Then $\lim_{\nu\to\infty} p^\nu=p^\varepsilon$. By the lower semi-continuity of $E(p)$, $\lim_{\nu\to\infty} x^\nu=x^\varepsilon$ for some sequence $\{x^\nu\}$ such that $x^\nu \in E(p^\nu)$ for all ν. Therefore there exists a positive integer N such that

$$(4) \qquad \max[0, x_i^N]+\varepsilon > \lambda^\varepsilon p_i^N \text{ for } i=1, 2, \cdots, m.$$

For $i>m$, $p_i^N=p_i^\varepsilon$. Since $p^N \geqq p^\varepsilon$, $x^N \in E(p^N)$ and $x^\varepsilon \in E(p^\varepsilon)$, this implies $x_i^N \geqq x_i^\varepsilon$ by (S). Using (3),

$$(5) \quad \max[0, x_i^N]+\varepsilon \geqq \max[0, x_i^\varepsilon]+\varepsilon=\lambda^\varepsilon p_i^\varepsilon=\lambda^\varepsilon p_i^N \text{ for } i=m+1, \cdots, n.$$

Combining (4) with (5), it has been shown that

$$\max[0, x^N]+\varepsilon e \geqq \lambda^\varepsilon p^N=\lambda\left(\frac{N}{N+1} p^N\right),$$

where $\lambda=(1+1/N)\lambda^\varepsilon$. On the other hand $x^N \in E(p^N)=E\left(\frac{N}{N+1} p^N\right)$ by (H), and $\frac{N}{N+1} p^N \in S$. Thus $\lambda \in M^\varepsilon$, contradicting to the definition of λ^ε.

4) $\max[0, x]=(\max[0, x_1], \max[0, x_2], \cdots, \max[0, x_n])$.

Since λ^ε is a nondecreasing function with respect to ε and bounded from below, there exists a λ^0 such that $\lim_{\nu\to\infty} \lambda^{1/\nu} = \lambda^0$. Corresponding to any ν, as already shown, there exist $p^{1/\nu}$ and $x^{1/\nu}$ such that

$$(6) \qquad p^{1/\nu} \in S,\ x^{1/\nu} \in E(p^{1/\nu}) \quad \text{and} \quad \max[0, x^{1/\nu}] + \frac{1}{\nu} e = \lambda^{1/\nu} p^{1/\nu}.$$

Then, since S and $E(S)$ are compact, we may assume without loss of generality that $\lim_{\nu\to\infty} p^{1/\nu} = p^* \in S$, $\lim_{\nu\to\infty} x^{1/\nu} = x^* \in E(S)$. By the upper semi-continuity of $E(p)$, we have also $x^* \in E(p^*)$.

Letting $\nu \to \infty$ in (6), we have $\max[0, x^*] = \lambda^0 p^*$.

This and (W) imply that

$$0 \geqq \lambda^0(p^*, x^*) = \sum_{i=1}^{n} \lambda^0 p_i^* x_i^* = \sum_{i=1}^{n} (\max[0, x_i^*])^2 \geqq 0.$$

Hence $x_i^* \leqq 0$ for all $i = 1, 2, \cdots, n$. Thus p^* is an equilibrium price vector. Q.E.D.