87. A Construction of Branching Markov Processes

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In order to construct a branching Markov process[1] we are able to take, roughly speaking, two ways: One is so-called a probabilistic way, that it is obtained by piecing out a given Markov process which we call the non-branching part by means of a given branching system, and the other is so-called an analytic way, that solving a fundamental equation determined by a given Markov process and a branching system,[1] it is obtained by constructing a semi-group on $\boldsymbol{C}(\boldsymbol{S})$ from the obtained solution. This paper is devoted to the construction of branching Markov process in a probabilistic way. The analytic construction will be given in a forthcoming paper.

1. Direct product of Markov processes. Let S be a compact Hausdorff space with a countable base and Δ be an extra point. Let $\{W, x_t, \mathscr{B}_t, \theta_t, P_x, x \in S\}$ be a right continuous strong Markov process on $S \cup \{\Delta\}$ with Δ as a death point.[2] We set $\zeta(w) = \text{int}\{t; x_t(w) = \Delta\}$ ($= +\infty$ if such t does not exist), then it is clear that $x_t(w) \in S$ if $t \in [0, \zeta(w))$ and $x_t(w) = \Delta$ if $t \in [\zeta(w), +\infty)$. We assume that

$$P_x[\zeta = t] = 0, \text{ for } t \geqq 0 \text{ and } x \in S.$$

Let $W^{(n)}$ be the n-fold product of W. The element of $W^{(n)}$ is denoted as $w' = (w^1, w^2, \cdots, w^n)$ where $w^j \in W, j = 1, 2, \cdots, n$, and put

$$x'_t(w') = (x_t(w^1), x_t(w^2), \cdots, x_t(w^n)), \tag{1.1}$$

and

$$\bar{\zeta}(w') = \min_{1 \leqq k \leqq n} \{\zeta(w^k)\}. \tag{1.2}$$

Now, we define

$$\bar{x}_t(w') = \begin{cases} \gamma[x'_t(w')], & \text{if } t < \bar{\zeta}(w'),^{3)} \\ \Delta &, \text{if } t \geqq \bar{\zeta}(w'), \end{cases} \tag{1.3}$$

and

$$\theta_t w' = (\theta_t w^1, \theta_t w^2, \cdots, \theta_t w^n). \tag{1.4}$$

Then it is easy to see that $\bar{x}_t(w')$ is a random variable defined

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1) In this paper we adopt the terminology and the notation used in [2], [3], and [4]. For the definition of branching Markov processes we refer to [2]. For the definition of fundamental equations of branching Markov processes we refer to [3].

2) i.e. i) W contains w_Δ such as $x_t(w_\Delta) = \Delta$ for every $t \geqq 0$ and ii) for every $w \in W$, $x_t(w) = \Delta$ implies $x_s(w) = \Delta$ for all $s \geqq t$. When it is necessary to introduce P_Δ, we take any probability measure P_Δ on $(W, \mathscr{B}_\infty)$ scuh that $P_\Delta[x_0(w) = \Delta] = 1$.

3) γ is the natural mapping from $S^{(n)}$ to S^n, cf. [2].

on $(W^{(n)}, \mathcal{B}^{(n)}=\bigotimes_{i=1}^{n} \mathcal{B}_\infty)$[4] taking values in $S^n \cup \{\Delta\}$ such that (i) $\bar{x}_t(w')=\Delta$ implies $\bar{x}_s(w')=\Delta$ for all $s \geqq t$ and (ii) $\bar{x}_t(\theta_s w')=\bar{x}_{t+s}(w')$. Let

(1.5) $$\overline{\mathfrak{N}}_t^{(n)}=\mathcal{B}\{\bar{x}_s(w');\ \forall s \leqq t\} \subset \mathcal{B}^{(n)}$$

be the smallest σ-field with respect to which $\bar{x}_s(w')$ is measurable for any $s \leqq t$.

Now, if we define $\bar{P}_{(x_1,x_2,\cdots,x_n)}$, $(x_j \in S, j=1, 2, \cdots, n)$ by

(1.6) $$\bar{P}_{(x_1,x_2,\cdots,x_n)}[A]=P_{x_1}\times P_{x_2}\times\cdots\times P_{x_n}[A],^{5)} \text{ for } A \in \overline{\mathfrak{N}}_\infty^{(n)}$$

then we have

Lemma 1.1. *For any* $A \in \overline{\mathfrak{N}}_\infty^{(n)}$,

$$\bar{P}_{(x_1,x_2,\cdots,x_n)}[A]=\bar{P}_{(\pi x_1,\pi x_2,\cdots \pi x_n)}[A],$$

where $(\pi x_1, \pi x_2, \cdots, \pi x_n)$ *may be any permutation of* $(x_1, x_2, \cdots, x_n)$.

Therefore for any $\boldsymbol{x} \in S^n$,

(1.7) $$\bar{P}_{\boldsymbol{x}}[A]=\bar{P}_{(x_1,x_2,\cdots,x_n)}[A], \qquad A \in \overline{\mathfrak{N}}_\infty^{(n)}$$

is well-defined, where $(x_1, x_2, \cdots, x_n) \in \boldsymbol{x}$.

Definition 1.1. The above defined system $\{W^{(n)}, \bar{x}_t(w'), \overline{\mathfrak{N}}_t^{(n)}, \bar{\zeta}, \theta_t, \bar{P}_x, \boldsymbol{x} \in S^n\}$ is said to be *the n-fold symmetric direct product*[6] of the given Markov process $\{W, x_t, \mathcal{B}_t, \zeta, \theta_t, P_x, x \in S\}$.

Theorem 1.1. *The n-fold symmetric direct product* $\{W^{(n)}, \bar{x}_t, \overline{\mathfrak{N}}_t^{(n)}, \bar{\zeta}, \theta_t, \bar{P}_x, \boldsymbol{x} \in S^n\}$ *of a given right continuous strong Markov process* $\{W, x_t, \mathcal{B}_t, \zeta, P_x, x \in S\}$ *is a right continuous strong Markov process. If* $x_t(w)$ *has the left limit, then* $\bar{x}_t(w')$ *also has the left limit.*

Our proof of Theorem 1.1 is based on Fubini's theorem and the following

Lemma 1.2. (i) *Let* $A \in \overline{\mathfrak{N}}_t^{(n)}$, *and* A_j *be the j-section of A, i.e., fixing* $w^1, w^2, \cdots, w^{j-1}, w^{j+1}, \cdots, w^n$, *we put* $A_j=\{w^j; (w^1, \cdots, w^n) \in A\}$. *Then* $A_j \in \mathcal{B}_t$.

(ii) *Let* $T(w')$ *be* $\overline{\mathfrak{N}}_t^{(n)}$*-Markov time. Then the j-section* $T^j(w)$ *of* $T(w')$ *which is defined, for fixed* $w^1, \cdots, w^{j-1}, w^{j+1}, \cdots, w^n$, *by*

$$T^j(w^j)=T(w')$$

is a $\mathcal{B}_t$*-Markov time.*

(iii) *Let T be an* $\overline{\mathfrak{N}}_t^{(n)}$*-Markov time and* $A \in \overline{\mathfrak{N}}_T^{(n)}$. *Then*

$$A_j \in \mathcal{B}_{T^j},$$

where A_j *and* T^j *are the j-section of A and T, respectively.*

Proposition 1.1. *If* $\{W, x_t, \mathcal{B}_t, \zeta, P_x, x \in S\}$ *is quasi-left continuous before* ζ, *then the symmetric direct product is also quasi-left continuous before* $\bar{\zeta}$.

Proposition 1.2. *If* $\{W, x_t, \mathcal{B}_t, \zeta, P_x, x \in S\}$ *satisfies Hunt's hypothesis (A)*[7] *and if* ζ *is non-accessible,*[8] *then the symmetric direct*

4) $\bigotimes_{i=1}^{n} \mathcal{B}_\infty$ is the n-fold product of $\mathcal{B}_\infty$.

5) $P_{x_1}\times P_{x_2}\times\cdots\times P_{x_n}$ is the product measure of $P_{x_1}, \cdots$, and P_{x_n}.

6) The n-fold direct product of different kinds of processes is similarly defined and the following results are valid for both.

7) Cf. Hunt [1].

8) i.e. totally inaccessible in the strong sense in the sense of Meyer [5].

product satisfies also the hypothesis (A).

2. To construct a non-branching part on S from a given Markov process.

Given a right continuous strong Markov process $\{W, x_t, \mathcal{B}_t, \zeta, \theta_t, P_x, x \in S\}$ on $S \cup \{\Delta\}$ with Δ a death point, let $X^{(n)} = \{W^{(n)}, \bar{x}_t, \mathcal{N}_t^{(n)}, \bar{\zeta}, \theta_t, \bar{P}_{\boldsymbol{x}}, \boldsymbol{x} \in S^n\}$ be its n-fold symmetric direct product. Then a non-branching part on $\boldsymbol{S}$ is defined as, roughly speaking, the infinite direct sum of $X^{(n)}$.

To be precise, take another extra point ∂ and an element w_∂ and put

$$(2.1) \qquad S^0 = \{\partial\}, \quad W^{(0)} = \{w_\partial\},$$

$$(2.2) \qquad \overline{W} = \bigcup_{n=0}^{\infty} W^{(n)},$$

$$(2.3) \qquad \bar{x}_t(\bar{w}) = \begin{cases} \bar{x}_t(w'), & \text{if } \bar{w} = w' \in W^{(n)}, n = 1, 2, \cdots, \\ \partial, & \text{if } \bar{w} = w_\partial \in W^{(0)}, \end{cases}$$

$$(2.4) \qquad \bar{\zeta}(\bar{w}) = \begin{cases} \bar{\zeta}(w'), & \text{if } \bar{w} = w' \in W^{(n)}, n = 1, 2, \cdots, \\ +\infty, & \text{if } \bar{w} = w_\partial \in W^{(0)}, \end{cases}$$

$$(2.5) \qquad \theta_t \bar{w} = \begin{cases} \theta_t w', & \text{if } \bar{w} = w' \in W^{(n)}, n = 1, 2, \cdots, \\ w_\partial, & \text{if } \bar{w} = w_\partial \in W^{(0)}, \end{cases}$$

and

(2.6) $\overline{\mathcal{N}}_t$, $(0 \leqq t \leqq +\infty)$ is the σ-field on $\overline{W}$ generated by $\mathcal{N}_t^{(n)}$, $n = 1, 2, \cdots$, and $\{w_\partial\}$.

Obviously, we have $\overline{\mathcal{N}}_{t|W^{(n)}} = \mathcal{N}_t^{(n)}$ and $\bar{x}_t(\bar{w}) = \Delta$ (resp. ∂) implies $\bar{x}_s(\bar{w}) = \Delta$ (resp. ∂) for every $s \geqq t$. Now we shall define a system of probability measures $\bar{P}_{\boldsymbol{x}}$, $\boldsymbol{x} \in \boldsymbol{S} = \bigcup_{n=0}^{\infty} S^n \cup \{\Delta\}$, on $(\overline{W}, \overline{\mathcal{N}}_\infty)$ by

$$(2.7) \qquad \begin{cases} \bar{P}_{\boldsymbol{x}}[A] = \bar{P}_{\boldsymbol{x}}[A \cap W^{(n)}], \text{ if } \boldsymbol{x} \in S^n, \\ \bar{P}_\partial[A] = \delta_\partial(A), \\ \bar{P}_\Delta \text{ is any probability measure on } (\overline{W}, \overline{\mathcal{N}}_\infty) \text{ such that} \\ P_\Delta[\bar{x}_0(\bar{w}) = \Delta] = 1. \end{cases}$$

Definition 2.1. Let $X^{(n)}$ be the symmetric direct product on S^n of a given Markov process on S. Then the above defined system $\bar{X} = \{\overline{W}, \bar{x}_t, \overline{\mathcal{N}}_t, \bar{\zeta}, \theta_t, P_{\boldsymbol{x}}, \boldsymbol{x} \in \boldsymbol{S}\}$ is said to be *the (infinite) direct sum of* $X^{(n)}$.

Theorem 2.1. *The above defined infinite direct sum is a right continuous strong Markov process on* $\boldsymbol{S}$, *and if* $x_t(w)$ *has the left limit, then* $\bar{x}_t(\bar{w})$ *has the left limit, if* $x_t(w)$ *is quasi-left continuous, then* $\bar{x}_t(\bar{w})$ *is also quasi-left continuous, and if* $x_t(w)$ *satisfies the Hunt's hypothesis (A) and* ζ *is non-accessible, then* $\bar{x}_t(\bar{w})$ *satisfies the hypothesis (A).*

The direct sum $\bar{X} = \{\overline{W}, \bar{x}_t, \overline{\mathcal{N}}_t, \bar{\zeta}, \theta_t, \bar{P}_{\boldsymbol{x}}, \boldsymbol{x} \in \boldsymbol{S}\}$ of the direct products is suited for a non-branching part of a branching Markov process.

3. To construct an instantaneous distribution from a given branching system. Given a branching system $\{q_n(x), \pi_n(x, d\boldsymbol{y}), n = 0, 2, 3, \cdots, +\infty\}$ (cf. [3]), let $\bar{X} = \{\overline{W}, \bar{x}_t, \overline{\mathcal{N}}_t, \bar{\zeta}, \theta_t, \bar{P}_{\boldsymbol{x}}, \boldsymbol{x} \in \boldsymbol{S}\}$ be the direct sum of the symmetric direct products of a given Markov process defined in the previous section. We assume that ζ satisfies

$$P_x[\exists x_{\zeta-} \in S]=1.$$

We define $\pi(x, d\boldsymbol{x})$ by

$$(3.1) \qquad \begin{cases} \pi(x, d\boldsymbol{x})=\sum_{n=0}^{\infty}{}^{*}\, q_n(x)\pi_n(x, d\boldsymbol{x}), \\ \pi(\partial, d\boldsymbol{x})=\delta_{\partial}(d\boldsymbol{x}), \\ \pi(\Delta, d\boldsymbol{x})=\delta_{\Delta}(d\boldsymbol{x}), \end{cases}$$

where $\sum_{n=0}^{\infty}{}^{*}$ denotes the sum over $n=0, 1, 2, \cdots, +\infty$.

Next we put, if $\bar{\zeta}(\bar{w})>0$ and $\bar{w} \in W^{(n)}$,

$$(3.2) \qquad \mu'(\bar{w}, d\boldsymbol{x}_1, d\boldsymbol{x}_2, \cdots, d\boldsymbol{x}_n)=\sum_{k=1}^{n} \chi_{\{\bar{\zeta}(\bar{w})=\zeta(w^k)<\infty\}}(\bar{w})\pi(x_{\zeta(w^k)-}(w^k), d\boldsymbol{x}_k) \times \prod_{j\neq k} \delta_{\{x_{\bar{\zeta}(\bar{w})}(w^j)\}}(d\boldsymbol{x}_j),$$

and if $\bar{\zeta}(\bar{w})=0$,

$$(3.3) \qquad \mu'(\bar{w}, d\boldsymbol{x})=\delta_{\Delta}(d\boldsymbol{x}).$$

Let γ be the mapping from $\boldsymbol{S}\times \boldsymbol{S}\times \cdots \times \boldsymbol{S}$ to $\boldsymbol{S}$, (cf. [2]), and define

$$(3.4) \qquad \begin{aligned} &\mu(\bar{w}, d\boldsymbol{x})=\mu'(\bar{w}, \gamma^{-1}(d\boldsymbol{x})), \text{ if } \bar{w} \in W^{(n)}, n=1, 2, \cdots, \\ &\mu(w_{\partial}, d\boldsymbol{x})=\delta_{\partial}(d\boldsymbol{x}), \end{aligned}$$

then, we have

Proposition 3.1. *The above defined kernel* $\mu(\bar{w}, d\boldsymbol{x})$ *on* $\overline{W}\times \boldsymbol{S}$ *is an instantaneous distribution.*[9]

4. To construct a branching Markov process from a given Markov process and a given branching system. Let $\bar{x}_t$ be the Markov process defined in §2 and $\mu(\bar{w}, d\boldsymbol{x})$ be the instantaneous distribution defined in §3. For $\bar{x}_t$ and μ, we can apply Theorem 1.1 of [4]. To be precise, the state space S in Theorem 1.1 of [4] is now $\bigcup_{n=0}^{\infty} S^n$ and $\bar{S}$ is $\boldsymbol{S}=\bigcup_{n=0}^{\infty} S^n \cup \{\Delta\}$. Thus we obtain the following

Theorem 4.1. *Let* $\{W, x_t, \mathcal{B}_t, \zeta, P_x, x \in S\}$ *be a right continuous strong Markov process on* $S\cup\{\Delta\}$ *with* Δ *a death point such that*

$$(4.1) \qquad P_x[\zeta=t]=0, \quad \text{for } x \in S \text{ and } t \geqq 0,$$

$$(4.2) \qquad P_x[\exists x_{\zeta-} \in S]=1, \quad \text{for } x \in S.$$

And let $\{q_n(x), \pi_n(x, d\boldsymbol{y}); n=0, 2, 3, \cdots, +\infty\}$ *be a branching system i.e.*

(a) $q_n(x)$ *is a non-negative Borel measurable function on* S *satisfying*

$$\sum_{n=0}^{\infty}{}^{*}\, q_n(x)=1, \quad \text{for any } x \in S.$$

(b) $\pi_n(x, d\boldsymbol{y})$ *is a Borel measurable function of* $x \in S$ *for fixed* $d\boldsymbol{y}$ *and a probability measure on* $(S^n, \mathcal{B}(S^n))$ *for fixed* $x \in S$, *where* $\mathcal{B}(S^n)$ *is the topological Borel field of* S^n.

Let us define $\pi(x, d\boldsymbol{y})$ *by* (3.1).

9) The definition of an instantaneous distribution is found in [4].

Then, there exists a unique (up to equivalence) right continuous strong Markov process $X=\{\tilde{\Omega}, X_t, \tilde{\mathscr{B}}_t, \tilde{\zeta}, \theta_t, \tilde{P}_x, \boldsymbol{x} \in \boldsymbol{S}\}$ *such that*

(4.3) $T_t\hat{f}(\boldsymbol{x})=\widehat{(T_t\hat{f})}|_S(\boldsymbol{x}), f \in \overline{\boldsymbol{B}}^*(S),$[10] *for* $\boldsymbol{x} \in \boldsymbol{S}$,

(4.4) $\tilde{E}_x[f(X_t); t<\tau]=T_t^0 f(x) \equiv E_x[f(x_t); t<\zeta]$, *for* $x \in S$,

(4.5) $\tilde{P}_x[X_\tau \in d\boldsymbol{y} \mid X_{\tau-}]=\pi(X_{\tau-}, d\boldsymbol{y})$, *a.s. on* $\{\tau<\infty\}$, *for* $x \in S$,

(4.6) $\tilde{P}_x[X_{\tau_\infty}=\Delta, \tau_\infty<\infty]=\tilde{P}_x[\tau_\infty<\infty]$,[11]

i.e. X_t *is a branching Markov process with the fundamental system* $\{T_t^0, \pi\}$ *satisfying the condition* (c. 2) *in* [2]. *Moreover, if* x_t *has the left limit,* X_t *has also the left limit at* $t<\tau_\infty$, *and if* x_t *is quasi-left continuous and* ζ *is non-accessible, then* X_t *is quasi-left continuous before* τ_∞.

The statements of the Theorem are verified, if we notice the way how the process X_t has been constructed and use the Theorem 1 of [2].

Remark. Under an additional condition

(4.7) $\sup_{x \in S} P_x[\zeta<\infty]=a<1$, or

(4.8) $\inf_{x \in S} P_x[\zeta>\varepsilon]>\delta$, for some $\varepsilon>0$ and $\delta>0$,

X_t is left continuous at τ_∞, when $\tau_\infty<\infty$.

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10) $T_t\hat{f}(\boldsymbol{x})=\tilde{E}_{\boldsymbol{x}}[\hat{f}(X_t)]$. For the definition of $\overline{\boldsymbol{B}}^*(S)$ and $\hat{f}$ we refer to [2], [3].

11) τ, τ_n, and τ_∞ are defined in the same way as in [2] and [4].