84. Some Applications of the Functional Representations of Normal Operators in Hilbert Spaces. XX

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Let $T(\lambda)$ be the same notation as that used in the preceding paper; that is, let $T(\lambda)$ be a function with singularities $\overline{\{\lambda_\nu\}}\cup[\bigcup_{j=1}^{n} D_j]$ such that the denumerably infinite set $\{\lambda_\nu\}$ denoting the set of poles of $T(\lambda)$ in the sense of the functional analysis is everywhere dense on a closed or an open rectifiable Jordan curve and that the mutually disjoint closed (connected) domains D_j ($j=1$ to n) have no point in common with the closure $\overline{\{\lambda_\nu\}}$ of $\{\lambda_\nu\}$ and lie in the disc $|\lambda|\leqq\sup_\nu|\lambda_\nu|$.

Theorem 56. Let the ordinary part of such a function $T(\lambda)$ as was stated above be a non-zero constant ξ; let c be an arbitrary finite complex number; let $\sigma=\sup_\nu|\lambda_\nu|$; let $n(\rho, c)$ be the number of c-points, with due count of multiplicity, of $T(\lambda)$ in the closed domain $\bar{\Delta}_\rho\{\lambda\colon \rho\leqq|\lambda|\leqq+\infty\}$ with $\sigma<\rho<+\infty$; let

$$N(\rho, c)=\int_\rho^{+\infty}\frac{n(r, c)-n(\infty, c)}{r}dr-n(\infty, c)\log\rho \quad (\sigma<\rho<+\infty),$$

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|T(\rho e^{-it})|\,dt \quad (\sigma<\rho<+\infty);$$

and let $M(\rho)=\max_{t\in[0,2\pi]}|T(\rho e^{-it})|$. Then $\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta$ is a decreasing function of s in the interval $|\xi|<s<M(\rho)$ for every ρ with $\sigma<\rho<+\infty$ and $m(\rho, \infty)$ is a decreasing convex function of $\log\rho$ for the interval $\sigma<\rho<+\infty$; moreover the equality

$$\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta=0$$

holds for every ρ with $\sigma<\rho<+\infty$ and every s with $M(\rho)\leqq s<+\infty$ and the equation $T(\lambda)-se^{i\theta}=0$ has no root in the domain $\{\lambda\colon \rho<|\lambda|<+\infty\}$ for every $\theta\in[0, 2\pi]$ and every s with $M(\rho)\leqq s<+\infty$.

Proof. Consider the function $f(\lambda)$ defined by

$$f(\lambda)=\begin{cases} T\left(\frac{1}{\lambda}\right)=\xi+\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^\mu & (\lambda\neq 0) \\ \xi \quad (\lambda=0) & \end{cases} \quad \left(0\leqq|\lambda|\leqq\frac{1}{\rho}, \sigma<\rho<+\infty\right),$$

where, as already shown before,

$$C_{-\mu}=\frac{1}{2\pi i}\int_{|\lambda|=\rho'}\frac{T(\lambda)}{\lambda^{-\mu+1}}d\lambda \quad (\sigma<\rho'<+\infty).$$

Then $f(\lambda)$ is regular in the closed domain $\overline{\mathfrak{D}}_{\rho^{-1}}\left\{\lambda: 0\leqq|\lambda|\leqq\frac{1}{\rho}\right\}$ with $\sigma<\rho<+\infty$. We next denote by $\tilde{n}(r, c)$ the number of c-points, with due count of multiplicity, of $f(\lambda)$ in the closed domain $\overline{\mathfrak{D}}_r\{\lambda: 0\leqq|\lambda|\leqq r\}$ with $0\leqq r\leqq\frac{1}{\rho}$ and set

$$\tilde{N}\left(\frac{1}{\rho}, c\right)=\int_0^{\frac{1}{\rho}}\frac{\tilde{n}(r, c)-\tilde{n}(0, c)}{r}dr+\tilde{n}(0, c)\log\frac{1}{\rho}.$$

If we now consider the function $g(\lambda)=a-\lambda$ for a non-zero complex constant a, then we find with the aid of Jensen's theorem that

$$\frac{1}{2\pi}\int_0^{2\pi}\log|a-se^{i\theta}|d\theta=\begin{cases}\log|a| & (|a|\geqq s)\\ \log|a|-\log\frac{|a|}{s} & (|a|<s).\end{cases}$$

Hence we have for every positive s

$$\frac{1}{2\pi}\int_0^{2\pi}\log|a-se^{i\theta}|d\theta=\overset{+}{\log}\frac{|a|}{s}+\log s. \tag{40}$$

On the other hand,

$$\log|f(0)-se^{i\theta}|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|f\left(\frac{1}{\rho}e^{it}\right)-se^{i\theta}\right|dt-\tilde{N}\left(\frac{1}{\rho}, se^{i\theta}\right)\quad(se^{i\theta}\neq\xi).$$

Here we integrate both sides with respect to θ and change the order of integration in the resulting double integral on the right-hand side. If, for any finite complex value c, all the c-points (repeated according to the respective orders) of $f(\lambda)$ in the domain $\left\{\lambda: 0<|\lambda|\leqq\frac{1}{\rho}\right\}$ are denoted by $a_1^{(c)}, a_2^{(c)}, \cdots, a^{(c)}_{\tilde{n}\left(\frac{1}{\rho},c\right)-\tilde{n}(0,c)}$, we have

$$\begin{aligned}\tilde{N}\left(\frac{1}{\rho}, c\right)&=\log\frac{\rho^{-\tilde{n}\left(\frac{1}{\rho},c\right)+\tilde{n}(0,c)}}{\left|a_1^{(c)}a_2^{(c)}\cdots a^{(c)}_{\tilde{n}\left(\frac{1}{\rho},c\right)-\tilde{n}(0,c)}\right|}+\tilde{n}(0, c)\log\frac{1}{\rho}\quad(\sigma<\rho<+\infty)\\ &=\log\frac{\left|\frac{1}{a_1^{(c)}}\frac{1}{a_2^{(c)}}\cdots\frac{1}{a^{(c)}_{n(\rho,c)-n(\infty,c)}}\right|}{\rho^{n(\rho,c)-n(\infty,c)}}-n(\infty, c)\log\rho\\ &=N(\rho, c).\end{aligned}$$

Accordingly the application of (40) to the result of the above-mentioned procedure enables us to attain to the equality

$$\overset{+}{\log}\frac{|\xi|}{s}+\log s=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{|T(\rho e^{-it})|}{s}dt+\log s-\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta,$$

so that

$$\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{|T(\rho e^{-it})|}{s}dt-\overset{+}{\log}\frac{|\xi|}{s}\quad(\sigma<\rho<+\infty). \tag{41}$$

Since, as will be seen from the principle of maximum modulus for $f(\lambda)$, $|\xi|<M(\rho)$, (41) implies that $\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta$ $(\sigma<\rho<+\infty)$ is a decreasing function of s in the interval $|\xi|<s<M(\rho)$ as we wished to prove. Since, however, $\tilde{n}(0, se^{i\theta})=n(\infty, se^{i\theta})=0$ for $se^{i\theta}\neq\xi$, it is clear that $N(\rho, se^{i\theta})\geqq 0$ for every ρ with $\sigma<\rho<+\infty$ and every finite $se^{i\theta}$ different from ξ and hence the right-hand side of (41) is never negative for every pair of such ρ and $se^{i\theta}$. In particular, we obtain the desired equality $\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta=0$ valid for every ρ with $\sigma<\rho<+\infty$ and every s with $M(\rho)\leqq s<+\infty$, as we were to prove. Since $N(\rho, se^{i\theta})\geqq 0$ for every $\theta\in[0, 2\pi]$, the final equality implies that the equation $T(\lambda)-se^{i\theta}=0$ has no root in the domain $D_\rho\{\lambda: \rho<|\lambda|<+\infty\}$ for every s with $M(\rho)\leqq s<+\infty$ and every $\theta\in[0, 2\pi]$: for otherwise there would exist uncountably many values of θ such that the inequality $\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta>0$ $(M(\rho)\leqq s<+\infty)$ would hold, contrary to fact, as can be verified immediately from the continuity based on the regularity of $T(\lambda)$ on D_ρ. If we next put $s=1$ in (41), then

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}N(\rho, e^{i\theta})d\theta+\overset{+}{\log}|\xi| \quad (\sigma<\rho<+\infty)$$

and so

$$\frac{dm(\rho, \infty)}{d\log\rho}=-\frac{1}{2\pi}\int_0^{2\pi}n(\rho, e^{i\theta})d\theta,$$

where $n(\rho, e^{i\theta})$ is a decreasing function of ρ in the interval $\sigma<\rho<+\infty$. As a result, it is easily verified that $m(\rho, \infty)$ is a decreasing convex function of $\log\rho$ for $\sigma<\rho<+\infty$.

Theorem 57. Let the ordinary part of the function $T(\lambda)$ stated before be a polynomial $\sum_{\mu=0}^{d}e_\mu\lambda^\mu$ of degree d; let σ be the same notation as before; and let $N(\rho, se^{i\theta})$, $m(\rho, \infty)$, and $M(\rho)$ be the notations associated with this $T(\lambda)$ in the same manners as those used to define $N(\rho, se^{i\theta})$, $m(\rho, \infty)$, and $M(\rho)$ in Theorem 56 respectively. Then (i) $|e_d|\leqq M(\rho)/\rho^d$ for every ρ with $\sigma<\rho<+\infty$; (ii) $\frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta$ is an increasing function of s in the interval $M(\rho)<s<+\infty$ for every ρ with $\sigma<\rho<+\infty$; (iii) there exists an uncountable set of values of $\theta\in[0, 2\pi]$ such that for any s greater than $|e_d|\rho^d$ with $\sigma<\rho<+\infty$ the equation $T(\lambda)-se^{i\theta}=0$ has at least one root in the domain $D_\rho\{\lambda: \rho<|\lambda|<+\infty\}$; (iv) $m(\rho, \infty)$ is a convex function of $\log\rho$ for $\sigma<\rho<+\infty$.

Proof. We now consider the function

$$\varphi(\lambda, se^{i\theta})=\begin{cases}(\rho\lambda)^d\left[T\left(\frac{1}{\lambda}\right)-se^{i\theta}\right] & (\lambda\neq 0)\\ e_d\rho^d & (\lambda=0),\end{cases}$$

where $\sigma<\rho<+\infty$ and $0\leqq|\lambda|\leqq\frac{1}{\rho}$. Then we have

$$\log|\varphi(0, se^{i\theta})|=\frac{1}{2\pi}\int_0^{2\pi}\log\left|\varphi\left(\frac{1}{\rho}e^{it}, se^{i\theta}\right)\right|dt-\hat{N}\left(\frac{1}{\rho}, 0\right),$$

where $\hat{N}\left(\frac{1}{\rho}, 0\right)$ is the notation associated with the number of zeros, with due count of multiplicity, of $\varphi(\lambda, se^{i\theta})$ in the domain $\bar{\mathfrak{D}}_{\rho^{-1}}\left\{\lambda: 0\leqq|\lambda|\leqq\frac{1}{\rho}\right\}$ by the same method as that used to define $\tilde{N}\left(\frac{1}{\rho}, c\right)$ for the function $f(\lambda)$ stated at the beginnig of the proof of Theorem 56. Since, moreover, $\hat{N}\left(\frac{1}{\rho}, 0\right)=N(\rho, se^{i\theta})$,

$$\log|e_d|\rho^d=\frac{1}{2\pi}\int_0^{2\pi}\log|T(\rho e^{-it})-se^{i\theta}|dt-N(\rho, se^{i\theta}).$$

By the same procedure as that used to establish (41) with the aid of (40), it is verified immediately from the final equality that

$$(42)\quad \frac{1}{2\pi}\int_0^{2\pi}N(\rho, se^{i\theta})d\theta=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{|T(\rho e^{-it})|}{s}dt+\log\frac{s}{|e_d|\rho^d}$$
$$(\sigma<\rho<+\infty).$$

Since $N(\rho, se^{i\theta})\geqq 0$, we can find by setting $s=M(\rho)$ in (42) that $|e_d|\rho^d\leqq M(\rho)$; and in addition, evidently the just established inequality and (42) imply that both (ii) and (iii) hold. If we next set $s=1$ in (42), then

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}N(\rho, e^{i\theta})d\theta+d\log\rho+\log|e_d|\quad(\sigma<\rho<+\infty)$$

and hence

$$(43)\quad \frac{dm(\rho, \infty)}{d\log\rho}=-\frac{1}{2\pi}\int_0^{2\pi}n(\rho, e^{i\theta})d\theta+d\quad(\sigma<\rho<+\infty),$$

where $n(\rho, e^{i\theta})$ denotes the number of $e^{i\theta}$-points, with due count of multiplicity, of $T(\lambda)$ in the domain $\bar{\Delta}_\rho\{\lambda: \rho\leqq|\lambda|\leqq+\infty\}$. Thus (iv) is shown in the same manner as in Theorem 56.

Theorem 58. Let $T(\lambda)$ and σ be the same notations as before, and let

$$m(\rho, c)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{1}{|T(\rho e^{-it})-c|}dt\quad(\sigma<\rho<+\infty, c\neq\infty).$$

If the ordinary part of $T(\lambda)$ is a non-zero complex constant or a polynomial in λ, then

$$\frac{1}{2\pi}\int_0^{2\pi}m(\rho, se^{i\theta})d\theta\leqq\log\frac{2}{s}\quad(\sigma<\rho<+\infty, 0<s\leqq 1).$$

Proof. We begin with the case where the ordinary part of $T(\lambda)$ is a non-zero complex constant ξ. Let $f(\lambda)$, $\tilde{n}(r, c)$, and $\tilde{N}\left(\frac{1}{\rho}, c\right)$ be the same notations as those defined at the beginning of the proof of Theorem 56. Then it is clear that $\tilde{n}(0, c)$ is not zero if and only if $c=\xi$ and that $\tilde{N}\left(\frac{1}{\rho}, \infty\right)=0$ $(\sigma<\rho<+\infty)$. If we now set

$$\tilde{m}\left(\frac{1}{\rho}, c\right)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}\frac{1}{\left|f\left(\frac{1}{\rho}e^{it}\right)-c\right|}dt & (c\neq\infty, \sigma<\rho<+\infty)\\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}\left|f\left(\frac{1}{\rho}e^{it}\right)\right|dt & (c=\infty, \sigma<\rho<+\infty)\end{cases}$$

and define $\varepsilon_j\left(\frac{1}{\rho}, c\right)$ $(j=1, 2)$ by

$$\tilde{m}\left(\frac{1}{\rho}, \infty\right)-\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\left|f\left(\frac{1}{\rho}e^{it}\right)-c\right|dt=\begin{cases}\varepsilon_1\left(\frac{1}{\rho}, c\right) & (c=\xi)\\ \varepsilon_2\left(\frac{1}{\rho}, c\right) & (c\neq\xi, \infty),\end{cases}$$

we can find from the inequality $\overset{+}{\log}\left|\sum_{\nu=1}^{p}\alpha_\nu\right|\leqq\sum_{\nu=1}^{p}\overset{+}{\log}|\alpha_\nu|+\log p$ valid for any complex numbers α_ν that $\left|\varepsilon_j\left(\frac{1}{\rho}, c\right)\right|\leqq\overset{+}{\log}|c|+\log 2$ for $j=1, 2$ and hence can analyze Nevanlinna's first fundamental theorem, as follows:

$$\tilde{m}\left(\frac{1}{\rho}, \infty\right)=\tilde{m}\left(\frac{1}{\rho}, c\right)+\tilde{N}\left(\frac{1}{\rho}, c\right)+K(\rho, c)\quad(\sigma<\rho<+\infty),$$

where

$$(44)\qquad K(\rho, c)=\begin{cases}0 & (c=\infty)\\ \log|C_{-1}|+\varepsilon_1\left(\frac{1}{\rho}, c\right) & (c=\xi, C_{-1}\neq 0)\\ \log|\xi-c|+\varepsilon_2\left(\frac{1}{\rho}, c\right) & (c\neq\xi, \infty).\end{cases}$$

In fact, for the special case $c=\xi$ we can attain to the second result of (44) by considering the auxiliary function

$$g(\lambda)=\begin{cases}\dfrac{f(\lambda)-\xi}{\rho\lambda}=\dfrac{1}{\rho}\displaystyle\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{\mu-1} & (C_{-1}\neq 0, \lambda\neq 0)\\ \dfrac{C_{-1}}{\rho} & (\lambda=0),\end{cases}$$

and the other two cases are trivial. Since, on the other hand, it is obvious that $\tilde{m}\left(\frac{1}{\rho}, c\right)=m(\rho, c)$ and $\tilde{N}\left(\frac{1}{\rho}, c\right)=N(\rho, c)$ both hold for every complex value c, finite or infinite, we obtain

$$(45)\qquad m(\rho, \infty)=m(\rho, c)+N(\rho, c)+K(\rho, c)\quad(c\neq\xi, \infty; \sigma<\rho<+\infty),$$

where $K(\rho, c)=\log|\xi-c|+\varepsilon_2\left(\frac{1}{\rho}, c\right)$. The application of (40) and (41) to (45) yields the relation

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi} m(\rho, se^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} \frac{|T(\rho e^{-it})|}{s} dt+\log s$$
$$+\frac{1}{2\pi}\int_0^{2\pi} \varepsilon_2\left(\frac{1}{\rho}, se^{i\theta}\right)d\theta$$

valid for $\sigma<\rho<+\infty$; and by utilizing $\log s=\overset{+}{\log} s-\overset{+}{\log}\frac{1}{s}$ and $\left|\varepsilon_2\left(\frac{1}{\rho}, se^{i\theta}\right)\right|\leqq \overset{+}{\log} s+\log 2$ to this result, we can easily show the validity of the inequality required in the statement of the theorem.

Suppose next that the ordinary part of $T(\lambda)$ is given by $\sum_{\mu=0}^{d} e_\mu \lambda^\mu$ where $e_d\neq 0$. We consider the function $f(\lambda)=T\left(\frac{1}{\lambda}\right)$ or the function $\varphi(\lambda, c)$ defined at the beginning of the proof of Theorem 57, according as $c=\infty$ or $c\neq\infty$. If we set

$$\varepsilon\left(\frac{1}{\rho}, c\right)=\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} |T(\rho e^{-it})| dt-\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} |T(\rho e^{-it})-c| dt$$
$$(c\neq\infty,\ \sigma<\rho<+\infty),$$

then, by reasoning exactly like that applied before, we can verify with the help of these auxiliary functions that

$$m(\rho, \infty)=m(\rho, c)+N(\rho, c)+K'(\rho, c), \tag{46}$$

where

$$K'(\rho, c)=\begin{cases}\log|e_d|+d\log\rho+\varepsilon\left(\frac{1}{\rho}, c\right) & (c\neq\infty)\\ d\log\rho \quad (c=\infty);\end{cases}$$

and here $\left|\varepsilon\left(\frac{1}{\rho}, c\right)\right|\leqq \overset{+}{\log}|c|+\log 2$. Since (46) and (42) enable us to conclude that

$$m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi} m(\rho, se^{i\theta})d\theta+\frac{1}{2\pi}\int_0^{2\pi} \overset{+}{\log} \frac{|T(\rho e^{-it})|}{s} dt$$
$$+\log s+\frac{1}{2\pi}\int_0^{2\pi} \varepsilon\left(\frac{1}{\rho}, se^{i\theta}\right)d\theta \quad (\sigma<\rho<+\infty),$$

the desired inequality in the statement of the theorem is established in the same manner as before.