80. On Axiom Systems of Propositional Calculi. XVII

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In his note [1], K. Iséki considers the equivalential calculus, given by S. Leśniewski [2], as the abstract set $\boldsymbol{M}=\langle M, \equiv\rangle$ which satisfies the following axioms:

1 $p\equiv r.\equiv.q\equiv p:\equiv.r\equiv q$,

2 $p\equiv.q\equiv r:\equiv:p\equiv q.\equiv r$.

By a variant of Lukasiewicz symbolism we can also write these axioms as

1 *EEEprEqpErq*,

2 *EEpEqrEEpqr*,

where E corresponds to the truth functor $\equiv$ (see A. N. Prior [3]).

The purpose of our paper is to present some theses of equivalential calculus. In this calculus, we use the rule of substitution and the rule of detatchment, i.e. α and $E_{\alpha\beta}$ imply β.

Using the rules of inference we can prove from the above axioms:

1 *EEEprEqpErq*,

2 *EEpEqrEEpqr*,

the following theses:

1 p/q, q/Eqp, r/Epq $*C2$ p/q, q/p, r/q—3,

3 *EEpqEqp*.

3 $p/EpEqr$, $q/EEpqr$ $*C2$—4,

4 *EEEpqrEpEqr*.

Having proved theses 3 and 4 we are now in a position to give a proof of the following

Theorem 1. *The equivalential calculus* $\boldsymbol{M}$ *is characterized by the following axioms*:

3 *EEpqEqp*,

4 *EEEpqrEpEqr*.

Proof. We shall use prooflines by J. Lukasiewicz for the proof of theses.

3 $p/EEpqr$, $q/EpEqr$ $*C4$—5,

5 *EEpEqrEEpqr*.

5 $p/EpEqr$, q/Epq $*C5$—6,

6 *EEEpEqrEpqr*.

3 $p/EEqErpEqr$, q/p $*C6$ p/q, q/r, r/p—7,

7 *EpEEqErpEqr*.

5 $q/EqErp$, r/Eqr $*C7$—8,

8 *EEpEqErpEqr*.

3 *p/EpEqErp*, *q/Eqr* *C8—9,

9 *EEqrEpEqErp*.

9 *p/Eqr*, *q/EpEqErp*, *r/EEpqErp* *C5 *r/Erp*—10,

10 *EEqrEEpEqErpEEEpqErpEqr*.

5 *p/Erq*, *q/EpErEqp*, *r/EEEprEqpErq* *C10 *q/r*, *r/q* —C9 *q/r*, *r/q*—11,

11 *EEEprEqpErq*.

In the theses above, theses 5 and 11 are the fundamental axioms of equivalential calculus. Now we have proved that the adopted axioms are equivalent to the axioms of equivalential calculus. Therefore we have completed the proof of Theorem 1.

Next we shall prove from the adopted axioms:

3 *EEpqEqp*,

4 *EEEpqrEpEqr*,

the following theses:

4 *p/Epq*, *q/Erp*, *r/Eqr* *C11 *q/r*, *r/q*—12,

12 *EEpqEErpEqr*.

12 *p/Epr*, *q/Erp*, *r/Eqr* *C3 *q/r*—13,

13 *EEEqrEprEErpEqr*.

3 *p/EEqrEpr*, *q/EErpEqr* *C13—14,

14 *EEErpEqrEEqrEpr*.

12 *p/EErpEqr*, *q/EEqrEpr*, *r/Epq* *C14—C12—15,

15 *EEEqrEprEpq*.

4 *p/Eqr*, *q/Epr*, *r/Epq* *C15—16,

16 *EEqrEEprEpq*.

3 *p/Eqr*, *q/EEprEpq* *C16—17,

17 *EEEprEpqEqr*.

4 *p/Epq*, *q/Epr*, *r/Erq* *C17 *q/r*, *r/q*—18,

18 *EEpqEEprErq*.

3 *p/EEqrEpr*, *q/Epq* *C15—19,

19 *EEpqEEqrEpr*.

19 *p/Erq*, *q/Eqr*, *r/Epr* *C3 *p/r*—20,

20 *EEEqrEprEErqEpr*.

18 *p/EEqrEpr*, *q/EErqEpr*, *r/Epq* *C20—C15—21,

21 *EEpqEErqEpr*.

16 *p/Epr*, *q/Eqr*, *r/Erq* *C3 *p/q*, *q/r*—22,

22 *EEEprErqEEprEqr*.

5 *p/Epq*, *q/Eqr*, *r/Epr* *C19—23,

23 *EEEpqEqrEpr*.

18 *p/EEprErq*, *q/EEprEqr*, *r/Epq* *C22—C23 *q/r*, *r/q* —24,

24 *EEpqEEprEqr*.

24 $p/ErEpq$, $q/EErpq$, $r/EEpqr$ $*C5$ p/r, q/p, r/q —$C3$ p/r, q/Epq—25,

25 $EEErpqEEpqr$.

24 $p/EErpq$, $q/EEpqr$, $r/ErEpq$ $*C25$—$C4$ p/r, q/p, r/q —26,

26 $EEEpqrErEpq$.

24 $p/EEpqr$, $q/ErEpq$, $r/EpEqr$ $*C26$—$C4$—27,

27 $EErEpqEpEqr$.

3 $p/ErEpq$, $q/EpEqr$ $*C27$—28,

28 $EEpEqrErEpq$.

3 $p/EEpqr$, $q/ErEpq$ $*C26$—29,

29 $EErEpqEEpqr$.

24 $p/ErEpq$, $q/EqErp$, $r/EEpqr$ $*C28$ p/r, q/p, r/q—$C29$—30,

30 $EEqErpEEpqr$.

3 $p/EqErp$, $q/EEpqr$ $*C30$—31,

31 $EEEpqrEqErp$.

5 $p/EqErp$, q/Epq $*C30$—32,

32 $EEEqErpEpqr$.

3 $p/EEqErpEpq$, q/r $*C32$—33,

33 $ErEEqErpEpq$.

21 p/Epq, $q/EErqEpr$, $r/ErEErqp$ $*C21$—$C27$ p/Erq, q/p—34,

34 $EEpqErEErqp$.

4 p/q, q/Erp, $r/EpEqr$ $*C28$ p/q, q/r, r/p—35,

35 $EqEErpEpEqr$.

3 p/q, $q/EErpEpEqr$ $*C35$—36,

36 $EEErpEpEqrq$.

4 p/Erp, $q/EpEqr$, r/q $*C36$—37,

37 $EErpEEpEqrq$.

3 p/Erp, $q/EEpEqrq$ $*C37$—38,

38 $EEEpEqrqErp$.

18 p/Erp, q/Epr, $r/EEpEqrq$ $*C3$ p/r, q/p—$C37$—39,

39 $EEEpEqrqEpr$.

4 $p/EpEqr$, r/Epr $*C39$—40,

40 $EEpEqrEqEpr$.

21 $p/ErEqp$, $q/EqEpr$, $r/EpEqr$ $*C27$ p/q, q/p—$C40$ —41,

41 $EErEqpEpEqr$.

4 p/r, q/Eqp, $r/EpEqr$ $*C41$—42,

42 $ErEEqpEpEqr$.

3 p/r, $q/EEqpEpEqr$ $*C42$—43,

43 $EEEqpEpEqrr$.

4 p/Eqp, $q/EpEqr$ $*C43—44$,

44 $EEqpEEpEqrr$.

3 p/Eqp, $q/EEpEqrr$ $*C44—45$,

45 $EEEpEqrrEqp$.

Each of theses 7, 12, 18, 21, 28, 31, 33, 34, 38, and 45 appears as an single axiom system of equivalential calculus in A. N. Prior ([3], p. 307), but the proofs are not given in it.

In the future notes we shall prove that each of these theses is equivalent to the fundamental axioms of equivalential calculus.

References

[1] K. Iséki: On axiom systems of propositional calculi. XV. Proc. Japan Acad.. **42**, 217-220 (1966).
[2] S. Leśniewski: Grundzüge eines neuen Systems der Grundlagen der Mathematik. Fund. Math., **14**, 1-81 (1929).
[3] A. N. Prior: Formal Logic. Oxford (1962).