74. On Some Applications of Selberg's Trace Formula

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1. **Definitions.** Let H be the upper half plane and Γ be a discrete subgroup of $G=SL(2, \boldsymbol{R})/\{\pm e\}$ acting on H such that $\Gamma\backslash H$ is compact (except in section 5, where Γ is the modular group). $d\tau=dxdy/y^2(\tau=x+iy \in H)$ is a G-invariant measure on H. We will consider an eigenvalue problem of Laplace operator

$$\varDelta=-y^2\Big(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\Big)$$

in $L^2(\Gamma\backslash H, d\tau)$. It has non negative discrete spectrum which we denote with Λ (or Λ_Γ).

A. Selberg proved the following important trace formula ([2]). If $h(r)$ is an even function satisfying certain analytical conditions, we have

$$(1)\quad \sum_{\lambda=\frac{1}{4}+r^2\in\varLambda} h(r)=\frac{A(D)}{2\pi}\int_{-\infty}^{\infty} r\frac{e^{\pi r}-e^{-\pi r}}{e^{\pi r}+e^{-\pi r}}h(r)dr+\int_{-\infty}^{\infty}E(r)h(r)dr$$
$$+2\sum_{i=1}^{\infty} n_i\sum_{k=1}^{\infty}\frac{\varepsilon_i}{a_i^{\frac{k}{2}}-a_i^{-\frac{k}{2}}}g(k\varepsilon_i),$$

where (i) $A(D)=\int_D d\tau$, D is a fundamental domain of Γ in H; (ii) $m_\beta(\beta=1, \cdots, s)$ are order of representatives of primitive elliptic classes in Γ and

$$E(r)=\frac{1}{2}\sum_{\beta=1}^{s}\sum_{k=1}^{m_\beta-1}\frac{1}{m_\beta\sin\dfrac{k\pi}{m_\beta}}\frac{e^{\pi r-2\pi r\frac{k}{m_\beta}}+e^{-\pi r+2\pi r\frac{k}{m_\beta}}}{e^{\pi r}+e^{-\pi r}};$$

(iii) $1<a_1<a_2<\cdots\uparrow\infty$ are norms, that is square of larger eigenvalues, of representatives $P_\alpha(\alpha=1, 2, \cdots)$ of primitive hyperbolic classes in Γ, $\varepsilon_i=\log a_i$ and n_i is the number of P_α whose norm is equal to a_i; (iv)

$$g(u)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-iru}h(r)dr.$$

In this note we will discuss some applications of this formula: a proof of an asymptotic formula for the eigenvalues (formula (2) in 2), a relation with Λ_Γ and Γ (Theorem in 3), a proof of an anounced result of I. M. Gelfand on a deformation of Γ (in 4) and an analogue of formula (2) for the modular group (in 5).

The details of the proof will be published elsewhere.

2. Asymptotic formula. Let $\alpha(\lambda)$ be the number of those elements of Λ which are smaller than λ. In (1) we put $h(r)=\exp\left\{-\left(\frac{1}{4}+r^2\right)t\right\}$, $t>0$. Then the left-hand side of (1) turns out to be $2\int_0^\infty \exp(-t\lambda)d\alpha(\lambda)$. The right-hand side is $A(D)/2\pi t+O(1)$, when $t\to 0$. Hence

$$\lim_{\lambda\to\infty}\alpha(\lambda)/\lambda=A(D)/4\pi. \tag{2}$$

This formula is a special case of the result anounced in [1].

3. Partial determination of Γ from Λ_Γ. Put

$$S_t=\sum_{i=1}^\infty n_i\sum_{k=1}^\infty \frac{\varepsilon_i}{a_i^{\frac{k}{2}}-a_i^{-\frac{k}{2}}}\exp\left(-\frac{k^2\varepsilon_i^2}{4t}\right).$$

Then we have

$$\lim_{t\to 0}\exp\left(\frac{\varepsilon_1^2}{4t}\right)S_t=n_1\frac{\varepsilon_1}{a_1^{\frac{1}{2}}-a_1^{-\frac{1}{2}}} \tag{3}$$

Theorem. *If $\Lambda_{\Gamma'}=\Lambda_{\Gamma''}$, then Γ' and Γ'' are isomorphic and $a_i'=a_i''$, $n_i'=n_i''$ $(i=1,2,\cdots)$, where a_i' means $a_{i\Gamma'}$ and so on.*

Outline of proof. The right-hand side of (1) for Γ' and Γ'' coincide with each other and $A(D')=A(D'')$ (from (2)).
Put $h(r)=r^{2n}\exp(-tr^2)$ and make $t\to 0$. Every moment of $E'(r)$ and $E''(r)$ coincides, so $E'(r)=E''(r)$. The formula

$$A(D)=2\pi(2g-2+\sum_{\beta=1}^{s}(1-1/m_\beta))$$

shows Γ' and Γ'' have the same genus. The first part of the theorem is thus established.

We have now $S_t'=S_t''$, if we put $h(r)=\exp(-tr^2)$ in (1). Repeated application of (3) will establish the last half of the theorem.

4. Deformation of Γ. Combining this Theorem with a result of A. Selberg (Lemma 4 in [3]), we can give a proof for a result of I. M. Gelfand anounced in [1], which states that a deformation of Γ (as a subgroup of $G=SL(2,\boldsymbol{R})$) which fixes Λ is a trivial one (see [3] for definitions).

5. Asymptotic formula for the modular group. Let Γ be modular group and consider an analogous eigenvalue problem. In this case $\Gamma\backslash H$ is not compact, so there is a continuous part in the spectrum. For the discrete part of the spectrum, we can show the formula of asymptotic distribution is same as (2). The method of proof is also analogous with the one in 2 and uses a modified trace formula ((3.12) in [2]).

References

[1] I. M. Gelfand: Proc. Int. Congress of Math., Stockholm (1962).
[2] A. Selberg: J. Ind. Math. Soc., **20** (1956).
[3] ——: Int. Colloquium on Function Theory. Bombay (1960).