73. The Plancherel Formula for the Lorentz Group of n-th Order

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Let $G(n)$ be the Lorentz group of n-th order, that is, the group of n-th order matrices g such that

$$ {}^t gJg = J, \ \det g = 1 \text{ and } g_{nn} \geqslant 1, \tag{1} $$

with

$$ J = \begin{pmatrix} 1 & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & 1 & \\ 0 & & & & -1 \end{pmatrix}. $$

In this note, we derive the Plancherel formula for $G(n)$. And we add some indications for the universal covering group $\tilde{G}(n)$ of $G(n)$ (when $n=3$). The formula has the same form as for $G(n)$ itself.

As is well known, for an infinitely differentiable function $f(g)$ on $G(n)$ with compact carrier and an irreducible unitary representation $g \rightarrow T_g$, the operator

$$ T_f = \int f(g) T_g d\mu(g) $$

has a trace in the corresponding representation space (here $d\mu(g)$ is a Haar measure on $G(n)$). This trace can be expressed by an invariant function $\pi(g)$ on $G(n)$ as

$$ Sp(T_f) = \int f(g) \pi(g) d\mu(g). $$

This function $\pi(g)$ is called the character of the representation $g \rightarrow T_g$.

The series of irreducible unitary representations which appear in the decomposition of a regular representation (i.e. principal series) was classified in [1] and their characters was obtained in [2]. Moreover the author proved recently that the reperesentations of the Lie algebra of $G(n)$ listed in [1] exhaust all algebraically irreducible ones which are induced by completely irreducible representations of $G(n)$. Therefore the results in [1] and [2] can be considered as the results concerning all infinitesimally equivalent classes of the completely irreducible representations of $G(n)$.

With the same notations in these papers, the principal series are the continuous series: $\mathfrak{D}_{(\alpha; i\rho)}$ and, in case n is odd, the discrete series: $\mathrm{D}^+_{(\alpha; p)}$ and $\mathrm{D}^-_{(\alpha; p)}$. For $\mathfrak{D}_{(\alpha; i\rho)}$, that trace is denoted by $Sp(T_f^\chi)$ with $\chi = (\alpha; i\rho)$, and the sum of the traces of $\mathrm{D}^+_{(\alpha; p)}$ and $\mathrm{D}^-_{(\alpha; p)}$ is denoted by $Sp(T_f^\nu)$ with $\nu=(p, \alpha)$.

1. Case of $n=2k+3$ $(k=0, 1, 2, \cdots)$.

For a representation $\mathfrak{D}_{(\alpha;i\rho)}$ in the continuous series, $i=\sqrt{-1}$, $\rho\in R$ (real number) and $\alpha=(n_1, n_2, \cdots, n_k)$ is a row of integers such that

$$0\leqslant n_1\leqslant n_2\leqslant \cdots \leqslant n_k.$$

We put $l_r=n_r+r-1/2$ $(1\leqslant r\leqslant k)$.

For representations $\mathrm{D}^+_{(\alpha;p)}$ and $\mathrm{D}^-_{(\alpha;p)}$ in the discrete series, p is an integer and $\alpha=(n_1, n_2, \cdots, n_k)$ is a row of integers such that, putting $n_0=p$,

$$0<n_0\leqslant n_1\leqslant n_2\leqslant \cdots \leqslant n_k.$$

Put $l_r=n_r+(r-1/2)$ $(0\leqslant r\leqslant k)$ as before.

Then the Plancherel formula has the form

$$\begin{aligned} cf(e)= &\sum_{0<l_1<l_2<\cdots<l_k}\int_{-\infty}^{\infty} iP(l_1, l_2, \cdots, l_k, -i\rho)\cdot th\pi\rho\ Sp(T_f^\nu)d\rho \\ &+\sum_{0<l_0<l_1<l_2<\cdots l_k} P(l_0, l_1, l_2, \cdots, l_k)Sp(T_f^\nu), \end{aligned} \qquad (2)$$

where e is the identy element of $G(n)$ and $P(X_1, X_2, \cdots, X_{k+1})$ is a polynomial of $X_1, X_2, \cdots, X_{k+1}$ corresponding to the product of all positive roots of Lie algebra of $G(n)$, that is,

$$P(X_1, X_2, \cdots, X_{k+1})=X_1X_2\cdots X_{k+1}\prod_{1\leqslant s<r\leqslant k+1}(X_r^2-X_s^2). \qquad (3)$$

The constant $c>0$ must be determined with respect to some normalization of Haar measure of $G(n)$.

The plancherel formula for $\tilde{G}(n)$ differ from the one for $G(n)$ only that the sums in (2) must be taken also for two-valued representations, that is, $\mathfrak{D}_{(\alpha;i\rho)}$, $\mathrm{D}^+_{(\alpha;p)}$, and $\mathrm{D}^-_{(\alpha;p)}$ (except the ones with $p=1/2$) for which all n_r in $\alpha=(n_1, n_2, \cdots, n_k)$ and p are half-integers satisfying respectively the inequalities indicated above $(p\neq 1/2)$.

As the colorary of this Plancherel formula, we know that all unitary representations $\mathrm{D}^+_{(\alpha;p)}$ and $\mathrm{D}^-_{(\alpha;p)}$ are square-summable and no other irreducible unitary representations of $G(n)$ is square-summable. But for the group $\tilde{G}(n)$, if $p=1/2$, $\mathrm{D}^+_{(\alpha;p)}$ and $\mathrm{D}^-_{(\alpha;p)}$ are not square-summable as for the case $n=3$.

2. Case of $n=2k+2$ $(k=1, 2, \cdots)$.

In this case, the derivation of the Plancherel formula are quite similar for complex classical groups, because there exists only one conjugate class of Cartan subgroup. But, for the completeness, we state here its explicite formula.

The parameter $(\alpha; i\rho)$ of a representation $\mathfrak{D}_{(\alpha;i\rho)}$ in the continuous series is such that $i=\sqrt{-1}$, $\rho\in \mathrm{R}$ and $\alpha=(n_1, n_2, \cdots, n_k)$ is a row of integers satisfying

$$|n_1|\leqslant n_2\leqslant \cdots \leqslant n_k.$$

Putting $l_r=n_r+(r-1)$ $(1\leqslant r\leqslant k)$, the Plancherel formula has the form

$$cf(e)=\sum_{|l_1|<l_2<\cdots<l_k}\int_{-\infty}^{\infty}P(l_1, l_2, \cdots, l_k, i\rho)Sp(T_f^{\chi})d\rho, \tag{4}$$

where $P(X_1, X_2, \cdots X_{k+1})$ is a polynomial corresponding to the product of all positive roots of Lie algebra of $G(n)$, that is,

$$P(X_1, X_2, \cdots, X_{k+1})=\prod_{1\leqslant s<r\leqslant k+1}(X_r^2-X_s^2). \tag{5}$$

The constant $c>0$ can be determined by the similar method employed in [3] for complex classical groups.

The plancherel formula for $\widetilde{G}(n)$ has the same form but the sum in (4) must be taken also for two-valued representations $\mathfrak{D}_{(\alpha;i\rho)}$.

3. For the case of odd n, $G(n)$ has two conjugate classes of Cartan subgroup. The method employed in this case by the author is essentially the same that is outlined by Harish-Chandra in [4].

Using Harish-Chandra's deep results in [4] and [5] concerning the function F_f on Cartan subgroups (constructed from f, see [4]), the problem is reduced to the deduction of some relations containing values of derived functions of F_f on hypersurfaces of their discontinuities. The reason is as follows. The traces $Sp(T_f^{\chi})$ and $Sp(T_f^{\nu})$ are expressed by certain sums of Fourier coefficients or Fourier-Laplace transformations of F_f. The value of f at e can be expressed as follows in two terms when we rewrite using the Fourier coefficients of F_f the right side of the following equality in [5]

$$cf(e)=\partial(\pi')F_f(e).$$

The first term cosists of the above mentioned sums of Fourier coefficients of F_f and therefore can be expressed by $Sp(T_f^{\chi})$ and $Sp(T_f^{\nu})$. The second term cosists of boundary values of F_f and its derived functions at singular elements of the Cartan subgroups. If we can prove that the second term is equal to zero, then we obtain the desired formula. For this, it needs some relations between these boundary values, containing the values of F_f and its derived functions on hypersurfaces of their discontinuity. We call these relations simply gap-relations.

The first method deducing the necessary gap-relations is very simple and has rather algebraic character. If we write down explicitely the fact that the characters of finite dimensional representations of $G(n)$ are eigendistributions of the Casimir operator of $G(n)$, we can obtain a gap-relation of the derived functions of first order of F_f. The necessary gap-relations of derived functions of higher order can be obtained from this one by the results in [5]. Also, they can be deduced by using another Laplace operators on $G(n)$ instead of the Casimir operator.

Another method of deducing necessary gap-relations is the direct calculation using the integral expressions of F_f as for the case

$n=3$ in [6].

Another method which can be employed to prove that the second term must be zero is the one used by Mr. K. Okamoto for the de Sittre group. This method needs not any gap-relation. From the boundedness of the numerators of the characters in [2] of $\mathfrak{D}_{(\alpha; i\rho)}$, $D^+_{(\alpha; p)}$, and $D^-_{(\alpha; p)}$, we can prove that the second term must be zero, by Riemann-Lebesgue theorem on Fourier series.

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