68. *Transformation of PGO into a Calculable Expression: Problem-Solving Machines. III*

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A theorem of elementary geometry can be expressed logically in the following way [1].

By a *term* is meant concatenated five digits such as 30110, which represents intuitively an object in elementary geometry such as a vertex of a triangle. By a *predicate letter* is meant concatenated three digits such as 710, which represents a relation between terms such as equality. Let R be a predicate letter. Let $A_1, A_2, \cdots, A_n$ be terms, respectively. Then we call $R(A_1, A_2, \cdots, A_n)$ an *atomic formula*, so that it shows that there is a relation R among terms $A_1, A_2, \cdots, A_n$. By a *formula* of elementary geometry is meant an expression built up from atomic formulas by use of logical symbols: $\neg, \cup, \cap, \rightarrow$. Each term may be regarded as a code assigned to an object of plane geometry and we call this code system PGO.

A problem-solving machine for elementary geometry has a memory to store theorems of elementary geometry in the form of PGO-expression. Given a problem in a natural sentence, the problem-solving machine translates it into a PGO-expression and retrieves the memory by use of standard form [2]. If retrieval is unsuccessful, the PGO-expression will be transformed into a computer calculable expression by use of analytical geometry. The transformation procedure will be described in the following way.

The last digit of each term is a parameter to distinguish one object from others.

Let Φ be a formula of PGO. Let $0000i$ be a code of a point occurring in Φ, where i is a digit of parameter. Then, with $0000i$ we correlate a pair of variables (x_i, y_i), where the parameter i of PGO acts as the suffix of the variable.

Let $1p00i$ be a code of a straight line or a segment occurring in Φ, where p is a factor to decide whether it is a straight line or a segment in accordance with 0 or 1. Then, with $1p00i$ we correlate two distinct pairs of variables (x_{i0}, y_{i0}) and (x_{i1}, y_{i1}).

Let $3pqri$ be a code associated with a triangle occurring in Φ,

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where p represents the attribute of a triangle, q represents one part of a triangle such as vertex and median and r acts as a parameter to distinguish the part represented by p from other parts, so that $30qri$, $31qri$, $32qri$, and $33qri$ denote one part of a general triangle, a regular triangle, an isosceles triangle and a right triangle, respectively. With $3pqri$ we correlate three distinct pairs of variables (x_{i0}, y_{i0}), (x_{i1}, y_{i1}), and (x_{i2}, y_{i2}). In particular, we associate in addition the following equations with $31qri$,

$$(x_{i0}-x_{i1})^2+(y_{i0}-y_{i1})^2=(x_{i1}-x_{i2})^2+(y_{i1}-y_{i2})^2 \\ =(x_{i2}-x_{i0})^2+(y_{i2}-y_{i0})^2,$$

with $32qri$,

$$(x_{i0}-x_{i2})^2+(y_{i0}-y_{i2})^2=(x_{i1}-x_{i2})^2+(y_{i1}-y_{i2})^2,$$

and with $33qri$,

$$(x_{i0}-x_{i1})(x_{i0}-x_{i2})+(y_{i0}-y_{i1})(y_{i0}-y_{i2})=0.$$

With other geometric figures, we can correlate pairs of variables and equations showing the attribute of the figure in the similar way mentioned above.

It is easily seen that each pair of variables represents a vertex. Let $3050i$ be an angle of a triangle. The following algebraic term is correlated with $305ri$:

$$\frac{|(x_{i1}-x_{i0})(y_{i2}-y_{i0})-(x_{i2}-x_{i0})(y_{i1}-y_{i0})|}{\sqrt{(x_{i1}-x_{i0})^2+(y_{i2}-y_{i0})^2}\sqrt{(x_{i2}-x_{i0})^2+(y_{i2}-y_{i0})^2}}.$$

These pairs of variables correlated with a term of PGO are distinct, so that $x_{ij}\neq x_{ik}$, $y_{ij}\neq y_{ik}$, $j=0, 1, 2$, $k=0, 1, 2$.

Next, we associate with each predicate letter of PGO suitable equations as shown in the following examples, denoting a predicate letter by parentheses [ ].

[000]; one point lies between two points.

$$000(00000, 00001, 00002). \tag{1}$$

The meaning of this formula (1) is that one point (x_0, y_0) lies between two points (x_1, y_1) and (x_2, y_2). With the formula (1), the following formula (2) is correlated.

$$\begin{aligned}&[(y_0-y_1)(x_2-x_0)=(y_2-y_0)(x_0-x_1)]\\ &\cap[((x_0-x_1)(x_2-x_0)>0)\cup((x_0-x_1)(x_2-x_1)=0)]\\ &\cap[((y_0-y_1)(y_2-y_0)>0)\cup((y_0-y_1)(y_2-y_1)=0)].\end{aligned} \tag{2}$$

[001]; divide internally.

$$001(00000, 00001, 00002). \tag{3}$$

The meaning of this formula (3) is that a point (x_0, y_0) divides two points (x_1, y_1) and (x_2, y_2) internally.

With the formula (3), the same formula as (2) is correlated.

[002]; divide externally.

$$002(00000, 00001, 00002). \tag{4}$$

The meaning of the formula (4) is that a point (x_0, y_0) divides two points (x_1, y_1) and (x_2, y_2) externally. With the formula (4), the following formula (5) is correlated.

$$\begin{aligned}(5)\qquad &[(y_0-y_1)(x_2-x_0)=(y_2-y_0)(x_2-x_1)]\\ &\cap[((x_0-x_1)(x_2-x_1)<0)\cup((x_0-x_1)(x_2-x_0)=0)]\\ &\cap[((y_0-y_1)(y_2-y_1)<0)\cup((y_0-y_1)(y_2-y_0)=0)].\end{aligned}$$

[003]; harmonic range.

The formula

(6) 003(00000, 00001, 00002, 00003)

represents that points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3) are in harmonic range, and the following formula (7) is correlated with (6).

$$\begin{aligned}(7)\qquad &[(x_1-x_0)(x_3-x_2)=(x_2-x_1)(x_3-x_0)]\\ &\cap[(x_3-x_0)(y_2-y_1)=(x_3-x_2)(y_1-y_0)]\\ &\cap[(y_1-y_0)(y_3-y_2)=(y_2-y_1)(y_3-y_0)]\\ &\cap[((x_1-x_0)(x_2-x_1)>0)\cup((x_1-x_0)(x_2-x_1)=0)]\\ &\cap[((y_1-y_0)(y_2-y_1)>0)\cup((y_1-y_0)(y_2-y_1)=0)]\\ &\cap[((x_3-x_0)(x_2-x_3)<0)\cup((x_3-x_0)(x_2-x_3)=0)]\\ &\cap[((y_3-y_0)(y_2-y_3)<0)\cup((y_3-y_0)(y_2-y_3)=0)].\end{aligned}$$

[004]; colinear.

The formula

(8) 004(00000, 00001, 00002)

expresses that three points (x_0, y_0), (x_1, y_1), and (x_2, y_2) are colinear. With the formula (8), the following formula (9) is correlated.

$$(9)\qquad \begin{vmatrix} x_0 & y_0 & 1\\ x_1 & y_1 & 1\\ x_2 & y_2 & 1\end{vmatrix}=0.$$

[010]; a straight line passes through a point.

The formula

(10) $010(0000i, 1000j)$

expresses that a straight line, $1000j$, passes through a point, $0000i$. With the formula (10), the following formula (11) is correlated.

$$(11)\qquad (y_{j0}-y_{j1})(x_i-x_{j1})=(y_i-y_{j1})(x_{j0}-x_{j1}).$$

[110]; two straight lines cross at a point.

The formula

(12) $110(0000i, 1000j, 1000k),\quad j<k$

expresses that two straight lines, $1000j$ and $1000k$, cross at a point, $0000i$. With the formula (12), the following formula (13) is correlated.

$$\begin{aligned}(13)\qquad &\neg\left[\begin{vmatrix} x_{j0}-x_{j1} & y_{j0}-y_{j1}\\ x_{k0}-x_{k1} & y_{k0}-y_{k1}\end{vmatrix}=0\right]\\ &\cap[(y_{j0}-y_{j1})(x_i-x_{j1})=(y_i-y_{j1})(x_{j0}-x_{j1})]\\ &\cap[(y_{k0}-y_{k1})(x_i-x_{k1})=(y_i-y_{k1})(x_{k0}-x_{k1})].\end{aligned}$$

[111]; perpendicular.

The formula

(14) $111(0000i, 1000j, 1000k), \quad j<k$

expresses that two straight lines, $1000j$ and $1000k$, are perpendicular at a point, $0000i$. With the formula (14), the following formula (15) is correlated.

$$\begin{aligned}&[(x_{j0}-x_{j1})(x_{k0}-x_{k1})+(y_{j0}-y_{j1})(y_{k0}-y_{k1})=0]\\ (15) \quad &\cap [(y_{j0}-y_{j1})(x_i-x_{j1})=(y_i-y_{j1})(x_{j0}-x_{j1})]\\ &\cap [(y_{k0}-y_{k1})(x_i-x_{k1})=(y_i-y_{k1})(x_{k0}-x_{k1})].\end{aligned}$$

[112]; parallel.

The formula

(16) $112(1000j, 1000k), \quad j<k$

expresses that two straight lines, $1000j$ and $1000k$, are parallel. With the formula (16), the following formula (17) is correlated.

$$(17) \quad \begin{vmatrix} x_{j0}-x_{j1} & y_{j0}-y_{j1} \\ x_{k0}-x_{k1} & y_{k0}-y_{k1} \end{vmatrix}=0.$$

[113]; concurrent.

The formula

(18) $113(1000i, 1000j, 1000k), \; i<j<k$

expresses that three straight lines, $1000i$, $1000j$, and $1000k$, are concurrent. With the formula (18), the following formula (19) is correlated.

$$(19) \quad \begin{vmatrix} x_{i0}-x_{i1} & y_{i0}-y_{i1} & x_{i0}y_{i1}-x_{i1}y_{i0} \\ x_{j0}-x_{j1} & y_{j0}-y_{j1} & x_{j0}y_{j1}-x_{j1}y_{j0} \\ x_{k0}-x_{k1} & y_{k0}-y_{k1} & x_{k0}y_{k1}-x_{k1}y_{k0} \end{vmatrix}=0.$$

Example. The theorem that the medians of a triangle are concurrent can be expressed in PGO as follows:

(20) 113(30330, 30340, 30350),

where 30330, 30340, and 30350 are medians of a triangle, respectively.

By A. Tarski [3] the same theorem is written as formula (21) by use of the following three relations:

$=$: the binary relation of identity.

B: the ternary relation of betweeness, so that $B(x, y, z)$ is to be read a point y is between two points x and z.

D: the quaternary relation of equidistance, so that $D(x, y; x', y')$ is to be read that a point x is as far from a point y as a point x' is from a point y',

where variables represent points, respectively.

$$\begin{aligned}&(\forall x)(\forall y)(\forall z)(\forall x')(\forall y')(\forall z')\{[\neg B(x, y, z)\\ &\cap \neg B(y, z, x)\cap \neg B(z, x, y)\cap B(x, y', z)\\ (21) \quad &\cap B(y, z', x)\cap B(z, x', y)\cap D(x, z'; z', y)\\ &\cap D(y, x'; x', z)\cap D(z, y'; y', x)]\\ &\rightarrow(\exists w)[B(x, w, x')\cap B(y, w, y')\cap B(z, w, z')]\}.\end{aligned}$$

The formula (20) of PGO is much simpler than the formula (21) and it is seen that a PGO-formula is suitable for memory.

The correlated formula with (20) is given as follows:

With the median, 30330, two pairs of variables (x_{00}, y_{00}) and $\left(\frac{x_{01}+x_{02}}{2}, \frac{y_{01}+y_{02}}{2}\right)$, with 30340 (x_{01}, y_{01}) and $\left(\frac{x_{02}+x_{00}}{2}, \frac{y_{02}+y_{00}}{2}\right)$ and with 30350 (x_{02}, y_{02}) and $\left(\frac{x_{00}+x_{01}}{2}, \frac{y_{00}+y_{01}}{2}\right)$ are correlated, respectively. For these pairs of variables the formula (19) is applied.

Thus, we can obtain the following formula (22).

$$\begin{vmatrix} 2x_{00}-x_{01}-x_{02} & 2y_{00}-y_{01}-y_{02} & x_{00}(y_{01}+y_{02})-y_{00}(x_{01}+x_{02}) \\ -x_{00}+2x_{01}-x_{02} & -y_{00}+2y_{01}-y_{02} & x_{01}(y_{02}+y_{00})-y_{01}(x_{02}+x_{00}) \\ -x_{00}-x_{01}+2x_{02} & -y_{00}-y_{01}+2y_{02} & x_{02}(y_{00}+y_{01})-y_{02}(y_{00}+y_{01}) \end{vmatrix} = 0. \tag{22}$$

The left hand side of the equation (22) can be easily seen to vanish identically.

References

[1] Kondô, M., and Murata, H.: On proof retrieval: problem-solving machines. I. Proc. Japan Acad., **41**, 254-259 (1965).

[2] ——: Standard form in PGO and transformation algorithm: problem-solving machines. II. Proc. Japan Acad., **41**, 355-359 (1965).

[3] Tarski, A.: A decision method for elementary algebra and geometry. Rand Report, R-109 (1948).