128. Existence and Uniqueness of Extensions of Volumes and the Operation of Completion of a Volume. I*)

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Introduction. Let R, Y be the space of reals and a Banach space respectively. The norm of elements in these spaces will be denoted by $|\ |$.

A nonempty family of sets V of an abstract space X will be called a *pre-ring* if for any two sets $A_1, A_2 \in V$ we have $A_1 \cap A_2 \in V$, and there exist disjoint sets $B_1, \cdots, B_k \in V$ such that $A_1 \backslash A_2 = B_1 \cup \cdots \cup B_k$.

A non-negative finite-valued function v on the pre-ring V will be called a *volume* if for every countable family of disjoint sets $A_t \in V (t \in T)$ such that $A = \bigcup_T A_t \in V$ we have $v(A) = \sum_T v(A_t)$.

In [1] has been presented a direct construction of the space $L(v, Y)$ of *Lebesgue-Bochner summable functions* and has been developed the theory of an integral of the form $\int u(f, d\mu)$. In the case when the bilinear form is given by $u(y, z) = zy$ for $y \in Y$, $z \in R$ and $\mu = v$ the above integral coincides with the classical Lebesgue-Bochner integral $\int f dv$.

All basic theorems concerning the algebraical and topological structures of the space $L(v, Y)$ have been proven without developing the theory of measure or the theory of measurable functions.

Basing the theory of integration on set functions defined on pre-rings it was possible in [2], [3] to develop the theory of *multilinear vectorial integration* and define *integral representations of multilinear continuous operators* on the space of Lebesgue-Bochner summable functions. It also permitted us to give new constructions of *Fubini's theorem* and to find its farther generalizations [4].

The theory of *Lebesgue-Bochner measurable functions* corresponding to the approach developed in [1] has been presented in [5]. The theory of *measure* has been obtained as a by-product of the theory of integration.

These results permitted us to simplify the theory of integration

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on *locally compact spaces* [6].

It was also shown in [7] that integration generated by *positive functionals* can be easily obtained from integration generated by volumes.

In this paper will be studied the operation of *completion of a volume v to a volume v_c*. The last volume is defined on the family V_c of all *v-summable sets*.

It will be also shown that any volume v on a pre-ring V has a *unique extension* to a volume on any pre-ring W such that $V \subset W \subset V_c$. These results play an important role in *characterizing volumes generating the same Lebesgue-Bochner integration* [8] as also in the theory of *extensions of vector valued set functions to measures* [9].

§1. The operation of completion of a volume. A *volume* v on a pre-ring V of X is called *complete* if the following three conditions are satisfied:

1. *If the family V forms a ring that is in addition to the axioms of a pre-ring it satisfies the condition $A \cup B \in V$ for any two sets $A, B \in V$,*

2. *For any increasing sequence of sets $A_n \in V$ such that the sequence $v(A_n)$ is bounded we have $A = \bigcup_n A_n \in V$,*

3. *If for some set $A \subset B \in V$ and $v(B)=0$ then $A \in V$.*

Let v be a volume and $L(v, R)$ the space of Lebesgue summable functions. Denote by V_c the family of all sets $A \subset X$ such that $\chi_A \in L(v, R)$. Define a new function v_c by means of the formula

$$v_c(A) = \int \chi_A dv \quad \text{for} \quad A \in V_c.$$

A set $A \subset X$ is called a *v-null-set* if for every $\varepsilon > 0$ there exists a countable family of sets $A_t \in V (t \in T)$ such that $A \subset \bigcup_T A_t$ and $\sum_T v(A_t) < \varepsilon$. Denote the family of all *v-null-sets* by N_v.

Theorem 1. *For every volume v the corresponding set function v_c represents a complete volume being an extension of the volume v.*

Proof. The function v_c is an extension of the function v according to Theorem 1-(5) of [1].

According to Theorem 4-(c) of [1] if $f, g \in L(v, R)$ then $h = f \cup g \in L(v, R)$, where

$$h(x) = \sup\{f(x), g(x)\} \quad \text{for} \quad x \in X.$$

Now if $A, B \in V_c$ then

$$\chi_{A \cup B} = \chi_A \cup \chi_B \in L(v, R)$$

and therefore $A \cup B \in V_c$. Similarly

$$\chi_{A \cap B} = -\{(-\chi_A) \cup (-\chi_B)\} \in L(v, R)$$

and therefore $A \cap B \in L(v, R)$.

From the identity

$$\chi_{A/B}=\chi_A-\chi_{A\cap B}$$

and from linearity of the spaces $L(v, R)$ we get $A/B\in V_c$. This proves that V_c is a *ring*.

Now take any increasing sequence of sets $A_n\in V_c$ such that the sequence

$$v_c(A_n)=\int\chi_{A_n}dv$$

is bounded. Since the sequence of functions χ_{A_n} is monotone with respect to relation less or equal almost everywhere therefore according to Theorem 4-(d) of [1] we have that there exists a function $g\in L(v, R)$ such that $\chi_{A_n}\to g$ a.e. Put $A=\bigcup A_n$. Notice that $\chi_{A_n}\to\chi_A$. Thus we have $g=\chi_A$ a.e. According to Theorem 1-(4) of [1] we get $\chi_A\in L(v, R)$ that is $A\in V_c$.

Now assume that $A\subset B\in V_c$ and $v_c(B)=0$. From Theorem 3 of [1] we have

$$v_c(B)=\int\chi_B dv=\|\chi_B\|_v$$

and therefore from Theorem 1-(2) of [1] there exists a null-set C such that $\chi_B(x)=0$ if $x\notin C$. This implies $B\subset C$ and $A\subset C$. That is we have $\chi_A=0$ a.e. Since 0-*function* is summable and according to Theorem 1-(4) of [1] a function equal almost everywhere to a summable function is summable we get $\chi_A\in L(v, R)$. That is $A\in V_c$.

From the identity

$$v_c(A)=\|\chi_A\|_v \quad \text{for} \quad A\in V_c$$

we see that the function v_c is non-negative.

To prove that the above function is a volume take any sequence of disjoint sets $A_n\in V_c$ and let $A=\bigcup_n A_n\in V_c$. Put $B_n=A_1\cup\cdots\cup A_n$. Since the sequence χ_{B_n} is convergent everywhere to the function χ_A and is dominated by the function χ_A therefore we have

$$\int\chi_{B_n}dv\to\int\chi_A dv.$$

This implies

$$v_c(A_1)+\cdots+v_c(A_n)+\cdots=v_c(A).$$

Thus we have proven that the function v_c is a *complete volume*.

Lemma 1. *Let v be a volume on V. If v is complete and $A_n\in V$ $(n=1, 2, \cdots)$ then $A=\bigcap_n A_n\in V$.*

Proof. Put $B_n=A_1\cap\cdots\cap A_n$ and notice that $A=\bigcap_n B_n$. The sequence B_n is increasing therefore the sequence $C_n=B_1\backslash B_n\in V$ is decreasing. Since $C_n\subset B_1$ therefore from monotonicity of a volume we get $v(C_n)\le v(B_1)$ for $n=1, 2, \cdots$. From the definition of a complete volume we get $C=\bigcup_n C_n\in V$. Thus $B_1\backslash C=\bigcap_n B_n=A\in V$.

Lemma 2. *Let v be a volume on V. If v is complete and $A\in N_v$ then $A\in V$ and $v(A)=0$.*

Proof. It follows from the definition of the family N_v that for every positive integer n there exists a sequence of sets $B_{nm} \in V$ such that

$$A \subset \bigcup_m B_{nm} \quad \text{and} \quad \sum_m v(B_{nm}) < 2^{-n}.$$

Reorder the double sequence of sets $\{B_{nm}\}$ into a single sequence $\{A_n\}$. We have

$$A \subset \bigcup_{n>m} A_n \quad \text{and} \quad \sum_n v(A_n) < 1.$$

Notice that

$$B_n = \bigcup_{m>n} A_m \in V \quad \text{and} \quad B = \bigcap_n B_n \in V$$

and therefore from monotonicity and sub-additivity of a volume we get

$$v(B) \leq v(B_n) \leq \sum_{m>n} v(A_m) \quad \text{for } n=1, 2, \cdots.$$

Thus we have $v(B)=0$. Since the volume v is complete we conclude $A \in V$.

Denote by S (*family of simple sets*) the family of all sets $A \subset X$ such that $\chi_A \in S(V, R)$. It is easy to see that *this family consists of all sets of the form* $A = A_1 \cup \cdots \cup A_k$ where $A_j \in V$ are disjoint sets.

Denote by S_δ the *family of all sets* of the form $A = \bigcap_n A_n$ where $A_n \in S$. Notice that $S_\delta \subset V_c$.

Denote by V_v the family of all sets of the form $A = \bigcup_n A_n$ where $A_n \in S_\delta$ is a sequence of sets such that the sequence of numbers $v_c(A_n)$ is bounded. We have $V_v \subset V_c$.

For any two sets $A, B \subset X$ define the *symmetric difference* operation by means of the formula $A \div B = (A/B) \cup (B/A)$. Any *ring of sets* with the *symmetric difference* operation forms a group.

Theorem 2. *Let v be a volume on a pre-ring V. A set A belongs to the domain V_c of v_c if and only if there exist sets $B \in N_v$ and $C \in V_v$ such that $A = B \div C$.*

Proof. Since $V_v \subset V_c$ and $N_v \subset V_c$ and the family V_c is a ring therefore we have $A = B \div C \in V_c$ for any $B \in N_v$, $C \in V_v$.

Conversely, take any set $A \in V_c$. Since $\chi_A \in L(v, R)$ therefore it follows from the definition of the space of summable functions that there exist a sequence of simple functions $s_n \in S(V, R)$ and a null-set $D \in N_v$ such that $s_n(x) \to \chi_A(x)$ if $x \notin D$. One may assume that the above sequence consists of non-negative functions, otherwise we would replace it by $s_n(x) \cup 0$.

Put $f_n(x) = \inf \{s_m(x) : m \geq n\}$. From the Monotone Convergence Theorem or from Theorem 4-(e) of [1] we get $f_n \in L(v, R)$.

Notice that the sequence of values $f_n(x)$ increasingly converges to the value $\chi_A(x)$ if $x \notin D$.

Put $C = \bigcup_n \{x \in X : f_n(x) > 0\}$. We see that $B = A \div C \subset D$ thus

$B \in N_v$.

Notice that $C = \bigcup_{np} C_{np}$ where $C_{np} = \left\{x \in X: f_n(x) \geq \frac{1}{p}\right\} (n, p = 1, 2, \cdots)$. We have also $C_{np} = \bigcap_{m \geq n} A_{mp}$ where $A_{mp} = \left\{x \in X: s_m(x) \geq \frac{1}{p}\right\}$. Since $A_{mp} \in S$ therefore $C_{np} \in S_\delta$.

Notice the inclusion $C_{np} \subset D \cup A$. Reorder the double sequence $\{C_{np}\}$ into a single sequence $\{D_n\}$. Put $C_n = D_1 \cup \cdots \cup D_n \in S$. Thus the sequence C_n of sets is increasing and $C_n \subset D \cup A$. Therefore $v_c(C_n) \leq v_c(A)$ for $n = 1, 2, \cdots$. Thus we get $C = \bigcup C_n \in V_v$.

Now since $B = A \div C$ we get $A = A \div 0 = A \div (C \div C) = (A \div C) \div C = B \div C$.

Theorem 3. *Let v be a volume on V. Then $v = v_c$ if and only if v is complete.*

Proof. If $v = v_c$ then according to Theorem 1 the volume v is complete.

Conversely let v be a complete volume. Since $v \subset v_c$ therefore to prove that both the volumes coincide it is enough to show that $A \in V_c$ implies $A \in V$.

Let $A = B \div C$ where $B \in V_v$ and $C \in N_v$. From Lemmas 1 and 2 we get $B \in V$ and $C \in V$. Since V is a ring therefore $A \in V$. Thus we have proven $v = v_c$.

References

[1] Bogdanowicz, W. M.: A generalization of the Lebesgue-Bochner-Stieltjes integral and a new approach to the theory of integration. Proc. Nat. Acad. Sc. USA, **53**, 492-498 (1965).

[2] ——: Multilinear Lebesgue-Bochner-Stieltjes integral, to appear in Trans. Amer. Math. Soc., for announcement of the results see Bull. Amer. Math. Soc., **72**, 232-236 (1966).

[3] ——: Integral representations of multilinear continuous operators from the space of Lebesgue-Bochner summable functions into any Banach space, to appear in Trans. Amer. Math. Soc., for announcement of the resuls see Bull. Amer. Math. Soc., **72**, 317-321 (1966).

[4] ——: Fubini theorems for generalized Lebesgue-Bochner-Stieltjes integral, to appear in Trans. Amer. Math. Soc., for announcement of the results see Proc. Japan Acad., **42** (1966), (supplement to **41**, 979-983 (1965)).

[5] ——: An approach to the theory of Lebesgue-Bochner measurable functions and to the theory of measure. Mathem. Annalen, **164**, 251-269 (1966).

[6] ——: An approach to the theory of integration and to the theory of Lebesgue-Bochner measurable functions on locally compact spaces. Mathem. Annalen (to appear).

[7] ——: An approach to the theory of integration generated by positive functionals and integral representations of linear continuous functionals on the space of vector valued continuous functions. Mathem. Annalen (to appear).

[8] ——: On volumes generating the same Lebesgue-Bochner integration (in press).
[9] ——: Vectorial integration and extensions of vector-valued set functions to measures (in press).