# 125. On Cauchy's Problem for a Linear System of Partial Differential Equations of First Order

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1. **Introduction.** In this note we shall show the existence and the uniqueness of the solution for a linear system of partial differential equations of the following form (1.1) satisfying the prescribed initial conditions (1.2):

$$(1.1) \qquad \frac{\partial u_\mu}{\partial t}=\sum_{\nu=1}^{k}\left\{\sum_{j=1}^{m}A_{\mu\nu j}(t,x)\frac{\partial u_\nu}{\partial x_j}+B_{\mu\nu}(t,x)u_\nu\right\}+f_\mu(t,x)$$

$$(1.2) \qquad u_\mu(0,x)=\varphi_\mu(x) \qquad (\mu=1,2,\cdots,k)$$

under some conditions on $A_{\mu\nu j}$, $B_{\mu\nu}$, $f_\mu$, and $\varphi_\mu$ which should be specified later (see [2]). We shall summarize here some notations and definitions. $R^m$ denotes the $m$-dimensional Euclidean space whose elements are denoted by $x=(x_1, x_2, \cdots, x_m)$, and $z=x+iy=(x_1+iy_1, x_2+iy_2, \cdots, x_m+iy_m)$ $(x, y\in R^m, i=\sqrt{-1})$ is an element of $m$-dimensional complex space $C^m$. For some positive $T$, $D(T)=\{(t,x); 0\leqq t\leqq T, x\in R^m\}$ and $\mathfrak{D}_\gamma(T)=\{(t,z); 0\leqq t\leqq T, z=x+iy\in C^m, |y_j|<\gamma, j=1,2,\cdots, m\}$ for some positive $\gamma$.

A function $f(t,x)$ which is $h$-time continuously differentiable with respect to $(t,x)$, is denoted by $f(t,x)\in C^h_{(t,x)}$, and that $f(t,x)$ which is analytic with respect to $x$ for each $t\in[0,T]$ is denoted by $f(t,x)\in A_{(x)}$.

For any positive constants $a$ and $b$, a function $f(t,x)$ belonging to $C_{(t,x)}$ on $D(T)$ and satisfying the inequality: $|f(t,x)|=Me^{ae^{b|x|}}$ on $D(T)$ for some positive constant $M$, is denoted by $f(t,x)\in F(a,b)$.

The method of the proof of the existence of the solution is essentially based on that of Prof. M. Nagumo [2]. The author wishes to express his deepest thanks to professor M. Nagumo for his kind advices and constant encouragement.

2. **Assumptions and Main Theorems. Assumptions.**

(I) The functions $A_{\mu\nu j}(t,x)$, $B_{\mu\nu}(t,x)$, $f_\mu(t,x)$ $(\mu,\nu=1,2,\cdots, k; j=1,2,\cdots,m)$ belong to $C_{(t,x)}$ on $D(T)$.

(II) The functions $A_{\mu\nu j}(t,x)$, $B_{\mu\nu}(t,x)$, $(\mu,\nu=1,\cdots,k;\ j=1,2,\cdots,m)$ belong to $A_{(x)}$ on $D(T)$ for each $t\in[0,T]$ and can be extended holomorphically with respect to $x$ to the complex domain $\mathfrak{D}_\gamma(T)$ on which they are continuous, and on $\mathfrak{D}_\gamma(T)$, $|A_{\mu\nu j}(t,z)|\leqq A$, $|B_{\mu\nu}(t,z)|\leqq B$ where $A$ and

$B$ are positive constants.

(III) The functions $f_\mu(t, x)$ $(\mu=1, 2, \cdots, k)$ belong to $A_{(x)}$ on $D(T)$ for each $t \in [0, T]$ and $\varphi_\mu(x)$ belong to $A_{(x)}$ on $R^m$. Moreover the functions $f_\mu(t, x)$, $\varphi_\mu(x)$ $(\mu=1, 2, \cdots, k)$ can be extended holomorphically with respect to $x$ to the complex domain $\mathfrak{D}_\gamma(T)$, on which they are continuous.

**Theorem 1.** *Under the assumptions* (I), (II), and (III), *there exist positive numbers* $T_1$ *and* $\gamma_1$ $(T_1 \leqq T, \gamma_1 < \gamma)$ *and a system of solutions* $u_\mu(t, z)$ *of* (1.1) *with the condition* (1.2) *which belong to* $C^1{}_{(t,z)}$ *on* $\mathfrak{D}_{\gamma_1}(T_1)$ *and to* $A_{(z)}$ *on* $\mathfrak{D}_{\gamma_1}(T_1)$ *for each* $t \in [0, T_1]$.

**Theorem 2.** *Under the assumptions* (I) *and* (II), *if* $u_\mu(t, x)$ *and* $v_\mu(t, x)$ $(\mu=1, 2, \cdots, k)$ *are continuously differentiable solutions of* (1.1) *on* $D(T)$ *satisfying the same initial conditions* (1.2) *and are contained in* $F(a, b)$ *for some constants* $a$ *and* $b$, *then* $u_\mu(t, x)= v_\mu(t, x)$ $(\mu=1, 2, \cdots, k)$ *on* $D(T)$.

**3. Preliminary lemmas. Lemma 1.** *Let* $f(z_1, z_2, \cdots, z_m)$ *be a holomorphic function in* $G(\delta)=\{z=x+iy;\ x_j, y_j \in R^1,\ |y_j|<\delta: j=1, 2, \cdots, m\}$ *which satisfies, for some positive constants* $M$, $\alpha$,

(3.1) $$|f(x_1+iy_1, x_2+iy_2, \cdots, x_m+iy_m)| \leqq M\rho^{-\alpha}$$

*where* $\rho=\delta-\max_j\{|y_j|\}$.

*Then in* $G(\delta)$ *the following inequalities hold for all* $j$: $(j=1, 2, \cdots, m)$.

(3.2) $$\left|\frac{\partial f}{\partial x_j}(x_1+iy_1, x_2+iy_2, \cdots, x_m+iy_m)\right| \leqq \frac{(1+\alpha)^{1+\alpha}}{\alpha^\alpha} M\rho^{-\alpha-1}.$$

**Proof.** For arbitrary $z^0 \in G(\delta)$ and any fixed $j$ we take a circle $C_j$ in the $z_j$-plane with radius $\frac{\rho}{1+\alpha}$ and with center $z_j^0$, where $\rho=\delta-\max_\nu\{|\mathfrak{Im} z_\nu^0|\}$. If $z_j \in C_j$, then $\delta-|\mathfrak{Im} z_j| \geqq \rho-\frac{\rho}{1+\alpha}$, and hence $|f(z)| \leqq (1+\alpha)^\alpha \alpha^{-\alpha} M\rho^{-\alpha}$. Therefore by Cauchy's integral formula we get the conclusion. Q.E.D.

In the proof of Theorem 1 and Theorem 2, we may assume for the initial values $\varphi_\mu(x)=0$, and then equations (1.1) with (1.2) are equivalent to the following functional equations:

(3.3) $$u_\mu(t, x)=\Phi_\mu[u(t, x)] \quad (\mu=1, 2, \cdots, k),$$

where

$$\Phi_\mu[u]=\sum_{\nu=1}^k\left\{\sum_{j=1}^m\int_0^t A_{\mu\nu j}(\tau, x)\frac{\partial u_\nu}{\partial x_j}(\tau, x)d\tau+\int_0^t B_{\mu\nu}(\tau, x)u_\nu(\tau, x)d\tau\right\}+\int_0^t f_\mu(\tau, x)d\tau.$$

Therefore to prove the Theorem 1 and Theorem 2, it is sufficient to prove the existence and the uniqueness of the solutions of (3.3).

**Lemma 2.** *Under the assumptions* (I), (II), and (III), *for arbitrary* $x^0 \in R^m$ *there exists a solution* $u(t, z) \in C^1_{(t,z)} \cap A_{(z)}$ *in any* *closed subdomain of* $\varDelta(x^0)$, *where*

$$\varDelta(x^0)=\{(t, x+iy);\ 0\leqq t\leqq T_1,\ |x_j-x_j^0|<R_1,\ |y_j|<R_1-L_1t\}$$

$$0<R_1<\text{Min}\left\{1,\ \gamma,\ \left(\frac{1+\alpha}{\alpha}\right)^{1+\alpha}(1-\alpha)mA/B\right\},\quad L_1=\frac{mkA}{\kappa}\left(\frac{1+\alpha}{\alpha}\right)^{1+\alpha}$$

*for any fixed* $\alpha$ *and* $\kappa$ *such that* $0<\alpha<1$, $0<\kappa<1$, *and*

$$T_1=\text{Min}\{T, R_1/L_1\}.$$

**Proof.** It is obvious that $g_\mu(t, z)\in C^1_{(t,z)}\cap A_{(z)}$ on $\mathfrak{D}_\gamma(T)$ implies $\Phi_\mu[g(t, z)]\in C^1_{(t,z)}\cap A_{(z)}$ on $\mathfrak{D}_\gamma(T)$. Now consider the sequence of functions $u_\mu^{(n)}(t, z)$ defined inductively as follows:

$$\begin{aligned} &u_\mu^{(0)}(t, z)=0 \\ &u_\mu^{(n+1)}(t, z)=\Phi_\mu[u^{(n)}(t, z)],\quad n=0, 1, 2, \cdots. \end{aligned} \tag{3.4}$$

Then from $u_\mu^{(0)}(t, z)\in C^1_{(t,z)}\cap A_{(z)}$ on $\mathfrak{D}_\gamma(T)$, it follows that $u_\mu^{(n+1)}(t, z)\in C_{(t,z)}\cap A_{(z)}$ on $\mathfrak{D}_\gamma(T)$ for all $n$.

Let $\Psi_\mu[u]=\Phi_\mu[u]-\int_0^t f_\mu(\tau, z)d\tau$, then

$$u_\mu^{(h+1)}-u_\mu^{(h)}=\Psi_\mu[u^{(h)}-u^{(h-1)}].$$

To demonstrate the convergence of the sequence $\{u_\mu^{(n)}(t, z)\}$, we consider the series:

$$\begin{aligned} u_\mu^{(n+1)}(t, z)&=\sum_{h=1}^{n}\{u_\mu^{(h+1)}(t, z)-u_\mu^{(h)}(t, z)\}+u_\mu^{(1)}(t, z) \\ &=\sum_{h=1}^{n}\Psi_\mu[u^{(h)}-u^{(h-1)}]+u_\mu^{(1)}(t, z). \end{aligned}$$

On the other hand, it is obvious that for given $\alpha$ $(0<\alpha<1)$ there exists a positive constant $M$ such that

$$|u_\mu^{(1)}-u_\mu^{(0)}|\leqq\int_0^t|f_\mu(\tau, z)|d\tau\leqq M\quad\text{in}\quad\varDelta(x^0)$$

where $\rho=(R_1-L_1t-\text{Max}\,|\mathfrak{J}_m z_j|)$, and hence we get

$$\int_0^t|u_\mu^{(1)}-u_\mu^{(0)}|\,d\tau\leqq\frac{M}{(1-\alpha)L_1}R_1^{1-\alpha}\quad\text{in}\quad\varDelta(x^0),$$

and from Lemma 1

$$\left|\int_0^t\frac{\partial(u_\nu^{(1)}-u_\nu^{(0)})}{\partial x_j}d\tau\right|\leqq\left(\frac{1+\alpha}{\alpha}\right)^{1+\alpha}\cdot\frac{M}{L_1}(\rho^{-\alpha}-R_1^{-\alpha}).$$

From the assumptions in Lemma we get the following:

$$|u_\mu^{(2)}-u_\mu^{(1)}|\leqq\kappa M\rho^{-\alpha}\quad\text{in}\quad\varDelta(x^0).$$

Thus we obtain inductively for all natural numbers $n$

$$|u_\mu^{(n+1)}-u_\mu^{(n)}|\leqq\kappa^n M\rho^{-\alpha}\quad\text{in}\quad\varDelta(x^0). \tag{3.5}$$

Therefore from (3.5) we obtain a function $u_\mu(t, z)$ which is the uniform limit function of $u_\mu^{(n)}(t, z)$ on any closed subdomain of $\varDelta(x^0)$. This shows that $u_\mu(t, z)\in C^1_{(t,z)}\cap A_{(z)}$ in $\varDelta(x^0)$ and $u_\mu(t, z)=\Phi_\mu[u(t, z)]$ in $\varDelta(x^0)$. Q.E.D.

**Remark 1.** From the above proof, we see that the solutions satisfy

$$|u_\mu(t, z)|\leqq\frac{M}{1-\kappa}\rho^{-\alpha}\quad\text{in}\quad\varDelta(x^0),\ \mu=1, 2, \cdots, k, \tag{3.6}$$

where $M=\sup_{(t,z)\in\Delta(x^0)}\{\rho^\alpha T_1|f_\mu(t,z)|\}$. We shall denote these solutions of (3.3) in $\Delta(x^0)$ constructed above by $u_\mu(t,z,x^0)$.

**4. Proof of Theorems. Proof of Theorem 1.** From the above Lemma 2, $u_\mu(t,z,x_0)\in C^1_{(t,z)}\cap A_{(z)}$ in $\Delta(x^0)$. For arbitrary $z\in\Delta(x^0)\cap\Delta(x^1)$, considering the function $v_\mu(t,z)=u_\mu(t,z,x^0)-u_\mu(t,z,x^1)$, we have $v_\mu(0,z)=0$ and $v_\mu(t,z)=\Psi_\mu[v(t,z)]$. If $\tilde{R}$ be a such positive number that

$$\Delta'=\left\{(t,z);\ 0\leqq t\leqq T_2,\left|x_j-\frac{x_j^0+x_j^1}{2}\right|<\tilde{R},\ |y_j|<\tilde{R}-L_1t\right\}\subset\Delta(x^0)\cap\Delta(x^1),$$

and $\tilde{\rho}=(\tilde{R}-L_1t-\operatorname{Max}_j|\Im m z_j|)$, $\tilde{M}=\sup_{\substack{(t,z)\in\Delta'\\ \mu=1,\cdots,k}}\{\tilde{\rho}^\alpha|v_\mu(t,z)|\}$, then we have the following inequalities:

$$|v_\mu(t,z)|=|\Psi_\mu[v(t,z)]|\leqq\kappa\tilde{M}\tilde{\rho}^{-\alpha}\quad\text{in}\quad\Delta'$$

as in the above proof of Lemma 2.

These facts show that $\tilde{M}\tilde{\rho}^{-\alpha}\leqq\kappa\tilde{M}\tilde{\rho}^{-\alpha}$ $(0<\kappa<1)$, that is to say $v_\mu(t,z)=0$ in $\Delta'$ $(\mu=1,\cdots,k)$. Hence we have by analytic continuation with respect to $z$, the solution $u_\mu(t,z)$ of (3.3) in $\mathfrak{D}_{\gamma_1}(T_1)$.

Q.E.D.

**Remark 2.** The Remark 1 and the Theorem 1 show that if $|f_\mu(t,z)|\leqq M\,exp\,(-ae^{b|x|})$ on $\mathfrak{D}_\gamma(T)$ for some positive constants $a$, $b$, and $M$, then for arbitrary $a'(<a)$ there exist $M'$ and $T_1$ such that the solutions of (3.3) satisfy

$$|u_\mu(t,x)|\leqq M'\ exp\,(-a'e^{b|x|})\quad\text{on}\quad D(T_1).$$

**Proof of Theorem 2.** Let

$$L_\mu[u]=\frac{\partial u_\mu}{\partial t}-\sum_{\nu=1}^k\left\{\sum_{j=1}^m A_{\mu\nu j}(t,x)\frac{\partial u_\nu}{\partial x_j}+B_{\mu\nu}(t,x)u_\nu\right\}$$

and for every $\sigma$, $(\sigma=1,2,\cdots,k)$

$$\tilde{L}_\mu^\sigma[u]=-\frac{\partial u_\mu}{\partial t}+\sum_{\nu=1}^k\left\{\sum_{j=1}^m\frac{\partial}{\partial x_j}[A_{\nu\mu j}(t,x)u_\nu]-B_{\nu\mu}(t,x)u_\nu\right\}$$
$$-e^{-ix\cdot\xi}\cdot exp\,\{-a'\cosh(b|x|)\}\cdot\delta_{\mu\sigma},\ \mu=1,2,\cdots,k,$$

where $a'$ is some positive constant such that for any given $\varepsilon>0$, $(a+\varepsilon)\,e^{b|x|}\leqq a'\cosh\{b|z|\}=\sum_{n=0}^\infty\frac{(b^2\sum_{\nu=1}^k z_\nu^2)^n}{(2n)!}$ on $\mathfrak{D}_\gamma(T)$ for sufficiently small positive $\gamma$, and $\delta_{\mu\sigma}$ is the Kronecker's delta.

The equations $\tilde{L}_\mu^\sigma[u]=0$ are of similar forms as equations considered in the Theorem 1, and considering $t$ in negative direction in the Theorem 1, we can conclude that there exist a positive $T_0(\leqq T_1)$ and the system of solutions $w_\mu(t,x)$ of $\tilde{L}_\mu^\sigma[u]=0$ in $D(T)$ with the initial condition $w_\mu(T,x)=0$ for any $T\in[0,T_0]$. Moreover from the Remark 1, we obtain the following inequalities:

$$(4.1)\quad |w_\mu(t,x)|\leqq M'\ exp\left\{-\left(a+\frac{\varepsilon}{2}\right)e^{b|x|}\right\}\quad\text{on}\quad D(T)\ (0<T\leqq T_0)$$

for some positive constant $M'$ depending on $\varepsilon$, if we choose the constant $a'$ appropriately for given $a$.

If $u$ and $v$ are the solutions of (1.1) with the condition (1.2), which belong to $F(a, b)$ for some positive $a$ and $b$, then the function $(u-v)$ satisfies $L_\mu[u-v]=0$, $(u_\mu-v_\mu)(0, x)=0$ and

(4.2) $|u_\mu(t, x)-v_\mu(t, x)| \leqq K\ exp(ae^{b|x|})$ on $D(T)$ ($\mu=1, 2, \cdots, k$)

for some positive constant $K$ and for any $T\in[0, T_0]$. Since

$$\sum_{\mu=1}^{k}\int_{D(T')}\int\{w_\mu L_\mu[u-v]-(u_\mu-v_\mu)\tilde{L}_\mu^\sigma[w]\}\,dxdt=0,$$

$$\int_0^t dt\cdot\int_{R^m} e^{-ix\cdot\xi}[(u_\sigma-v_\sigma)\,exp\{-a'\,cosh(b|x|)\}]dx=0$$

for any $\xi$ in $R^m$. Thus for any $\xi\in R^m$ and $t\in[0, T_0]$,

$$\int_{R^m} e^{-ix\cdot\xi}[(u_\sigma-v_\sigma)\,exp\{-a'\cdot cosh(b|x|)\}]dx=0. \tag{4.3}$$

Since $|(u_\sigma-v_\sigma)\,exp\{-a'\,cosh(b|x|)\}|\leqq exp\left\{-\frac{\varepsilon}{2}e^{b|x|}\right\}$, (4,3) shows that the Fourier transform of the integrable continuous function $(u_\sigma-v_\sigma)\,exp\{-a'\,cosh(b|x|)\}$ vanishes identically on $R^m$ for each $t\in[0, T_0]$. And since $exp\{-a'\,cosh(b|x|)\}\neq 0$ in $R^m$, $u_\sigma(t, x)-v_\sigma(t, x)=0$ on $D(T_0)$.

Now if there exists a $T'\in[0, T]$ for which holds $u_\mu(T', x)-v_\mu(T', x)\neq 0$ in $R^m$ for some $\mu$, let $T_2$ be the infimum of such $T'$, then $u_\mu(T', x)=v_\mu(T', x)$ on $D(T_2)$. In this case taking $T_2'$, $T_3$ such that $T_3-T_2\leqq T_0$ and $T_2'<T_2<T_3$, repeating the above argument for the interval $[T_2', T_3]$, we get $u_\mu(t, x)=v_\mu(t, x)$ for $(t, x)\in\{D(T_3)-D(T_2')\}=\{(t, x); T_2'<t\leqq T_3, x\in R^m\}$. This constradicts the assumption of the existence of $T'$ given above, and we get the conclusion

$u_\mu(t, x)=v_\mu(t, x)$ in $D(T)$ for every $\mu$. Q.E.D.

## References

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[2] M. Nagumo: Über das Anfangswertproblem Partieller Differentialgleichungen. Japanese Jour. of Math., **18**, 41-47 (1942).