122. *Non-Connection Methods for Some Connection Geometries based on Canonical Equations of Hamiltonian Types of II-Geodesic Curves*

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(Comm. by Zyoiti SUETUNA, M.J.A., June 13, 1966)

In [3], I established *non-connection methods* for linear connections *in the Large* bringing respective geometries to the "Erlanger Programm", the transformation group parameters being adequate functions of the (local) coordinates and in [4] he extended them further *doubly* to the case, where transformation group parameters are adequate functions of the (local) coordinates (x) as well as of $(\dot{x}, \ddot{x}, \cdots, \overset{(M)}{x})$, $(\dot{x}=dx/dt$, etc.; $t=$curve parameter). In [5], [6], and [8], M. Kurita studied the Finsler spaces by means of the canonical equations of Hamiltonian types. In this note, I will, being suggested by his means, establish the following geometries based on canonical equations of Hamiltonian types of the II-*geodesic curves* in my sense: (I) (Doubly) extended affine geometry, (II) (Doubly) extended Euclidean geometry, (III) Other 20 (doubly) extended geometries indicated on p. 247 of [14], (IV) Geometry of Finsler-Craig-Synge-Kawaguchi spaces, all based on canonical equations of Hamiltonian types of II-geodesic curves in the present author's sense. (IV) is a detailed exposition of the n-dimensional case of Art. 4 of [1].

I. (Doubly) Extended affine geometry based on canonical equations of Hamiltonian types of II-geodesic curves. **I.1.** *A new method of treatment of* II-*geodesic curves based on canonical equations of Hamiltonian types.* Consider

(I.1) $$\omega \overset{\text{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})dx^\mu, \quad (\lambda, \mu, \cdots=1, 2, \cdots, n),$$

which is *global* in the differentiable manifold $M=\bigcup_\alpha U_\alpha$ of class C^ν($\nu=$positive integer or ∞ or ω), where the open subset U_α is the domain of the local coordinates (x), since (I.1) is written in an invariant form.

Let $x^\lambda=x^\lambda(t)$ be a parametrized curve, where t is the canonical parameter ([14], Art. 12; [15], Art. 14). Set

(I.2) $$d\xi \overset{\text{def}}{=} \omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\dot{x}^\mu dt,$$

(I.3) $$L=\omega_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\dot{x}^\mu=p_\mu \dot{q}^\mu, \quad (q^\mu=x^\mu).$$

Then the Lagrangian equations for the extremal problem

(I.4) $$\delta\int Ldt=0$$

become

(I.5) $$\partial L/\partial x^{\mu}-d(\partial L/\partial\dot{x}^{\mu}-d(\partial L/\partial\ddot{x}^{\mu})/dt+\cdots+(-1)^{M-1}d^{M-1}(\partial L/\partial\overset{(M)}{x}{}^{\mu})/dt^{M-1})/dt=0.$$

Set

(I.6) $$p_{\mu}\overset{\text{def}}{=}\partial L/\partial\dot{q}^{\mu}-d(\partial L/\partial\ddot{q}^{\mu})/dt+\cdots+(-1)^{M-1}d^{M-1}(\partial L/\partial\overset{(M)}{q}{}^{\mu})/dt^{M-1},$$

for (I.3) anew, then (I.5) and (I.3) gives

(I.7) $$\dot{p}_{\mu}=\partial L/\partial q^{\mu},\ \dot{q}_{\mu}=\partial L/\partial p^{\mu},$$

forming Lagrangian canonical equations, and we have

$$\delta L=\delta p_{\mu}\dot{q}^{\mu}+p_{\mu}\delta\dot{q}^{\mu}=\dot{p}_{\mu}\delta t\dot{q}^{\mu}+p_{\mu}\delta\dot{q}^{\mu}=\dot{p}_{\mu}\delta q^{\mu}+p_{\mu}\delta\dot{q}^{\mu},$$

(I.8) $$\delta L=\delta(p_{\mu}\dot{q}^{\mu})+(\dot{p}_{\mu}\delta q^{\mu}-\dot{q}^{\mu}\delta p_{\mu})$$

and consequently for

(I.9) $$H\overset{\text{def}}{=}(p_{\mu}\dot{q}^{\mu})-L,$$

we have

(I.10) $$\delta H=\delta\{(p_{\mu}\dot{q}^{\mu})-L\}=\dot{q}^{\mu}\delta p_{\mu}-\dot{p}_{\mu}\delta q^{\mu},$$

whence follows the *canonical equations of Hamiltonian types*

(I.11) $$dq^{\mu}/dt=\partial H/\partial p_{\mu},\ dp_{\mu}/dt=-\partial H/\partial q^{\mu},\quad (dH/dt=0).$$

The curves represented by (I.6), (I.7) or by (I.11) will be called the II-*geodesic curves corresponding to* $\omega_{\mu}(x,\dot{x},\cdots,\overset{(M)}{x})$, which are extremals of (I.4).

Take n constants a^{l}, $(l=1,2,\cdots,n)$ not all equal to 0 and set

(I.12) $$L^{l}\overset{\text{def}}{=}a^{l}L,$$

so that

(I.13) $$\omega^{l}\overset{\text{def}}{=}a^{l}\omega,$$

(I.14) $$\omega_{\mu}^{l}(x,\dot{x},\cdots,\overset{(M)}{x})\overset{\text{def}}{=}a^{l}\omega_{\mu}(x,\dot{x},\cdots,\overset{(M)}{x}),$$

(I.15) $$H^{l}\overset{\text{def}}{=}a^{l}H=(a^{l}p_{\mu}\dot{q}^{\mu})-L^{l}=(p_{\mu}^{l}\dot{q}^{\mu})-L^{l},$$

(I.16) $$\delta H^{l}=\delta\{(p_{\mu}^{l}\dot{q}^{\mu})-L^{l}\}=\dot{q}^{\mu}\delta p_{\mu}^{l}-\dot{p}_{\mu}^{l}\delta q^{\mu},$$

(I.17) $$dq^{\mu}/dt=\partial H^{l}/\partial p_{\mu}^{l},\ (l\text{: not summed}),\ dp_{\mu}^{l}/dt=-\partial H^{l}/\partial q^{\mu},$$

(I.18) $$\xi^{l}\overset{\text{def}}{=}a^{l}\xi,\quad d\xi^{l}=\omega^{l}=a^{l}d\xi=a^{l}\omega.$$

The (I.17) as well as

(I.19) $$\begin{cases}p_{\mu}^{l}=\partial L^{l}/\partial\dot{q}^{\mu}-d(\partial L^{l}/\partial\ddot{q}^{\mu})/dt+\cdots+(-1)^{M-1}d^{M-1}(\partial L^{l}/\partial\overset{(M)}{q}{}^{\mu})/dt^{M-1}:\\ \dot{p}_{\mu}^{l}=\partial L^{l}/\partial q^{\mu},\ \dot{q}_{\mu}=\partial L/\partial p^{\mu},\quad (\text{cf. (I.6), (I.7)})\end{cases}$$

are *other systems of canonical equations of the* II-*geodesic curves.* p_{μ}^{l}, $\dot{p}_{\mu}^{l}$, L^{l}, and H^{l} are *components* of p_{μ}, $\dot{p}_{\mu}$, L, and H respectively. *The* II-*geodesic curves in the present author's sense coincide with the previous ones in* [4] as will be shown as follows. From (I.8), we obtain

(I.20) $$dL=d(p_{\mu}\dot{q}^{\mu})$$

and from (I.9):

(I.21) $$dL=d(p_{\mu}\dot{q}^{\mu})-dH,$$

so that we have

(I.22) $$dH/dt=0.$$

From (I.2), we obtain

(I.23) $$d\xi^l=L^l dt=a^l L dt=p^l_\mu dq^\mu,$$

so that

(I.24)

$$\xi^l=\int p^l_\mu dq^\mu=p^l_\mu q^\mu-\int q^\mu dp^l_\mu=p^l_\mu q^\mu-\int dp^l_\mu\int dq^\mu=p^l_\mu q^\mu-\iint(dp^l_\mu dq^\mu),$$

the condition for that the repeated integral may be converted into the double integral, i.e., that the integrand is continuous, being evidently satisfied. Now

(I.25) $$d^2\xi^l/dt^2=(dp^l_\mu/dt)(dq^\mu/dt)+p^l_\mu(d^2q^\mu/dt^2).$$

Since both terms on the right-hand side are written in invariant forms, if we take a transformation $\bar{q}^\mu=\bar{q}^\mu(q)$ such that $d^2\bar{q}^\mu/dt^2=0$, from (I.25), we must have

(I.26) $$dp^l_\mu d\bar{q}^\mu=0,$$

in which case (I.24) becomes of the form

$$\xi^l=p^l_\mu\bar{q}^\mu+p^l_0,\quad (p^l_0=\text{const.}).$$

Writing h, $\bar{\xi}$, ξ, and a for μ, ξ, $\bar{q}$, and p respectively, we obtain the formulas of *(doubly) extended affine transformation of the present author* ([4], (2.6), p. 872; [3], (3.2), *p*. 63):

(I.27) $$\bar{\xi}^l=a^l_h(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\xi^h+a^l_0,\quad (|a^l_h(\xi)|\neq 0)$$

accompanied by

(I.28) $$d\bar{\xi}^l=a^l_h(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})d\xi^h,$$

(I.29) $$da^l_h(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})d\xi^h=0,\quad (\text{cf. (I.26)}),$$

along the II-*geodesic line-elements.*

From (I.27) and (I.28), we obtain the *necessary condition*

(I.30) $$da^l_h(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})\xi^h=0$$

for the II-*geodesic line-elements.*

The $\bar{\xi}^l$ and the ξ^l will be called the II-*geodesic parallel coordinates.*

Setting

(I.31) $$ds\overset{\text{def}}{=}d\xi=Ldt,$$

from (I.23), we obtain

(I.32) $$\xi^l=a^l(s-s_0),\quad d\xi^l=L^l dt=a^l ds.$$

Since $d\bar{\xi}^l=\bar{a}^l ds$, $d\xi^l=a^l ds$, from (I.28), we see that a^l undergo the transformation

(I.33) $$\bar{a}^l=a^l_h(\xi,\dot{\xi},\cdots,\overset{(M)}{\xi})a^h=a^l_h(\xi,a,0,\cdots,0)a^h,$$

where $\bar{a}^l$ are const. on summation with respect to h.

The (I.32) shows us that *the* II-*geodesic curves behave as for meet and join like straight lines.* The s may be called the *affine length.*

I.2. *(Doubly) Extended affine geometry.* That *the totality of the (doubly) extended affine transformations* (I.27) *forms a group* may be shown by utilizing (I.30) quite as in p.64 of [3]. This group will be called the *(doubly) extended affine group* and the geometry under it the *(doubly) extended affine geometry.*

I.3. *The relation of the present method with that of* [4]. Since

(I.34) $$\omega^l = a^l\omega = p^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})dx^\mu,$$

we can show by straight forward calculation the identity:

(I.35) $$d^2\xi^l/ds^2 = d(\omega^l/ds)/ds \equiv p^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})\{d^2x^\mu/ds^2 + \Lambda^\lambda_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})(dx^\mu/ds)(dx^\nu/ds)\},$$

where the parameter $\Lambda^\lambda_{\mu\nu}$ of teleparallelism for $p^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})$ are defined by

(I.36) $$dp^l_\mu/ds - \Lambda^\lambda_{\mu\nu}p^l_\lambda(dx^\nu/ds) = 0, \mid dp^\lambda_l/ds + \Lambda^\lambda_{\mu\nu}p^\mu_l(dx^\nu/ds) = 0,$$

the p^λ_l being defined by

(I.37) $$p^l_\mu p^\lambda_l = \delta^\lambda_\mu \Longleftrightarrow p^l_\lambda p^\lambda_h = \delta^l_h$$

for p^l_μ, the δ's being Kronecker deltas.

Thus *the present method is equivalent to that of* [4].

I.4. *Another procedure.* Since we have (I.32), if we start with ξ^l in place of x^λ, (I.34) becomes

(I.38) $$\omega^l = a^l\omega = p^l_h(\xi, a, 0, \cdots, 0)d\xi^h$$

and thus our theory reduces to that of *(simply) extended geometry but for that n arbitrary parameters* (a^l) *appear in addition.*

II. (Doubly) Extended Euclidean geometry based on canonical equations of Hamiltonian types of II-geodesic curves.

II.1. *(Doubly) Extended Euclidean geometry based on canonical equations of Hamiltonian types of* II-*geodesic curves.* When the fundamental quadratic form of the (doubly) extended Euclidean geometry is

(II.1) $$ds^2 = g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x})dx^\mu dx^\nu,$$

it is always expressible in the form

(II.2) $$ds^2 = \omega^l\omega^l,$$

where

(II.3) $$\omega^l = \omega^l_\mu(x, \dot{x}, \cdots, \overset{(M)}{x})dx^\mu,$$

but for undergoing (doubly) extended orthogonal transformations.

If we adopt (II.2) for (I.1), the results of I holds still and (I.13) gives

(II.4) $$\omega^2 = \omega^l\omega^l = ds^2 = (a^la^l)\omega^2,$$

so that the condition

(II.5) $$a^la^l = 1$$

accompanies and (I.12) and (I.15) give

(II.6) $H^2 = H^l H^l$, (II.7) $L^2 = L^l L^l$, $dL^2 = dL^l dL^l$.

The (I.31) and the (I.32) show us that

(II.8) $$ds^2 = L^2 dt^2 = L^l L^l dt^2 = d\xi^l d\xi^l = \omega^l \omega^l,$$

(II.9) $$\omega^l = d\xi^l = L^l dt = a^l ds, \quad (a^l a^l = 1),$$

(II.10) $$d\xi = \omega = ds.$$

The (doubly) extended affine group becomes in this case the (*doubly*) *extended Euclidean group* and the (doubly) extended affine geometry the (*doubly*) *extended Euclidean geometry* [4]. The (II.3) shows us further that

(II.11) $$g_{\mu\nu} = \omega^l_\mu \omega^l_\nu.$$

In this way, we see that the present method leads us to *the* (*doubly*) *extended Euclidean geometry.*

II.2. *Another procedure.* If we take the view-point of I.4, *our theory reduces to that of* (*simply*) *extended Euclidean geometry but for that n arbitrary parameters* (a^l), ($a^l a^l = 1$) *appear in addition.*

III. Other (Doubly) extended geometries based on canonical equations of Hamiltonian types of II-geodesic curves. III.1. *Other* (*Doubly*) *extended geometries based on canonical equations of Hamiltonian types of* II-*geodesic curves. All other* (*doubly*) *extended geometries corresponding to the branches enlisted on p.* 247 *of* [14] *may be treated similarly* (*Mutatis mutandis*) *by means of canonical equations of Hamiltonian types of* II-*geodesic curves.*

IV. Geometry of Finsler-Craig-Synge-Kawaguchi spaces based on canonical equations of Hamiltonian types of II-geodesic curves. IV.1. *Finsler-Craig-Synge-Kawaguchi spaces.* These spaces are based on a certain integral

(IV.1) $$s = \int_{t_0}^{t_1} F(x, x', \cdots, \overset{(M)}{x}) dt, \quad (x' = dx/dt, \text{ etc.})$$

satisfying the so-called Zermero's conditions (cf. [13]). *The Kawaguchi space is reducible to the Finsler space having n transformation parameters* (a^l) *in addition* by transforming the coordinates (x^λ) to II-geodesic rectangular coordinates (ξ^l) of the present author in the base differentiable manifold (cf. (I.4)), so that $dx^\lambda/ds = a^\lambda$. Now for the Finsler space corresponding to

(IV.2) $$ds^2 = F^2(x, \dot{x})(ds/dt)^2(dt)^2,$$

where F is of degree one in $\dot{x} = dx/ds$, we have

(IV.3) $ds^2 = g_{\mu\nu}(x, \dot{x}) dx^\mu dx^\nu$, (IV.4) $g_{\mu\nu}(x, \dot{x}) = \frac{1}{2}(\partial^2 F(x, \dot{x})/\partial \dot{x}^\mu \partial \dot{x}^\nu)$.

The (IV.3) is always reducible to the form

(IV.5) $$ds^2 = \omega^l \omega^l, \quad (\omega^l = \omega^l_\mu(x, \dot{x}) dx^\mu)$$

but for undergoing (*doubly*) *extended orthogonal transformations.*

If we take (IV.5) for (II.8), *our theory of* II *gives a geometry of the Finsler-Craig-Synge-Kawaguchi spaces.*

IV.2. *Another procedure.* Another procedure is to adopt the metric tensor ([13], p. 724, $^*g_{ij}$):

$$g_{\mu\nu}(x, \dot{x}, \cdots, \overset{(M)}{x}) = MF^2 F_{(M)\mu} F_{(M)\nu} + \overset{M}{\mathfrak{C}}_\mu \overset{M}{\mathfrak{C}}_\nu + {}^*\overset{1}{\mathfrak{C}}_\mu {}^*\overset{1}{\mathfrak{C}}_\nu, \quad (F = F(x, \dot{x}, \cdots, \overset{(M)}{x})). \tag{IV.6}$$

The ds^2 is always expressible in the form (IV.5) and thus *our theory of* III *applies to the case of Finsler-Craig-Synge-Kawaguchi spaces.*

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