165. Some Applications of the Functional-Representations of Normal Operators in Hilbert Spaces. XXII

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Throughout the present paper we again treat the function $U(\lambda)\in\mathfrak{F}^*$ stated in Definition B and Theorem 60 of the preceding one; that is, $U(\lambda)$ is a function belonging to $\mathfrak{F}^*$ such that any point λ_ν of the denumerably infinite bounded set $\{\lambda_\nu\}_{\nu=1,2,3,\dots}$ assigned arbitrarily is an essential singularity of $U(\lambda)$ in the sense of the functional analysis, that the mutually disjoint closed domains $D_j(j=1, 2, 3, \cdots, n)$ with $\overline{\{\lambda_\nu\}}\cap\left[\bigcup_{j=1}^{n} D_j\right]=\phi$, assigned arbitrarily, form the sets of singularities of $U(\lambda)$ in the sense stated in Definition B and lie on the disc $|\lambda|\leqq\sup_\nu|\lambda_\nu|$, and that $U(\lambda)$ is regular in the complex λ-plane $\{\lambda: |\lambda|<+\infty\}$ except for $\overline{\{\lambda_\nu\}}\cup\left[\bigcup_{j=2}^{n} D_j\right]$.

Theorem 62. Let $\{\lambda_\nu\}$ be everywhere dense on a closed or an open rectifiable Jordan curve Γ; let the ordinary part of the function $U(\lambda)\in\mathfrak{F}^*$ stated above be a non-zero constant ξ; let c be an arbitrary complex number, finite or infinite; let $\sigma=\sup_\nu|\lambda_\nu|$; let $n(\rho, c)$ be the number (counted according to the respective multiplicities) of c-points of $U(\lambda)$ in the closed domain $\overline{\Delta}_\rho\{\lambda: \rho\leqq|\lambda|\leqq+\infty\}$ with $\sigma<\rho<+\infty$; let $\bar{n}(\rho, c)$ be the number of distinct c-points of $U(\lambda)$ in $\overline{\Delta}_\rho$; let

$$m(\rho, c)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}\frac{1}{|U(\rho e^{-it})-c|}dt & (c\neq\infty)\\ \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}|U(\rho e^{-it})|\,dt & (c=\infty),\end{cases}$$

$$N(\rho, c)=\int_\rho^{+\infty}\frac{n(r, c)-n(\infty, c)}{r}dr-n(\infty, c)\log\rho,$$

and

$$\bar{N}(\rho, c)=\int_\rho^{+\infty}\frac{\bar{n}(r, c)-\bar{n}(\infty, c)}{r}dr-\bar{n}(\infty, c)\log\rho$$

for any ρ with $\sigma<\rho<+\infty$; and let

$$\delta(c)=1-\varlimsup_{\rho\to\sigma+0}\frac{N(\rho, c)}{m(\rho, \infty)},$$

$$\Theta(c)=1-\varlimsup_{\rho\to\sigma+0}\frac{\bar{N}(\rho, c)}{m(\rho, \infty)},$$

$$\theta(c)=\lim_{\rho\to\sigma+0}\frac{N(\rho,c)-\overline{N}(\rho,c)}{m(\rho,\infty)}.$$

If

$$\text{(A)}\qquad \overline{\lim_{\rho\to\sigma+0}}\frac{m(\rho,\infty)}{\log\{1/(\rho-\sigma)\}}=+\infty,$$

then the set of values c for which $\Theta(c)>0$ is countable; and on summing over all such values c, the chain of inequalities $\sum_c\{\delta(c)+\theta(c)\}\leqq\sum_c\Theta(c)\leqq 2$ holds.

Proof. We now consider the function $f(\lambda)=U\left(\frac{1}{\lambda}\right)$ in the closed domain $\overline{\mathfrak{D}}_{\rho^{-1}}\left\{\lambda\colon 0\leqq|\lambda|\leqq\frac{1}{\rho}\right\}$ where $\sigma<\rho<+\infty$. Then $f(\lambda)$ is regular in $\overline{\mathfrak{D}}_{\rho^{-1}}$ and is expressible in the form

$$f(\lambda)=\xi+\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{\mu}\qquad\left(0\leqq|\lambda|\leqq\frac{1}{\rho},\ \sigma<\rho<+\infty\right),$$

where the second member on the right is essentially an infinite series such that the sum-function of the first and second principal parts of $U(\lambda)$ is given by $\sum_{\mu=1}^{\infty}C_{-\mu}\lambda^{-\mu}$ in the domain $\{\lambda\colon\sigma<|\lambda|\leqq+\infty\}$. On the assumption that c is any complex number, finite or infinite, we write $\tilde{n}(r,c)$ for the number of c-points of $f(\lambda)$ in the closed domain $\overline{\mathfrak{D}}_r\{\lambda\colon 0\leqq|\lambda|\leqq r\}$ where $r\leqq\frac{1}{\rho}$, c-points of order p being counted p times, and set

$$\tilde{N}\left(\frac{1}{\rho},c\right)=\int_0^{\frac{1}{\rho}}\frac{\tilde{n}(r,c)-\tilde{n}(0,c)}{r}dr+\tilde{n}(0,c)\log\frac{1}{\rho}\qquad(\sigma<\rho<+\infty),$$

where $\tilde{n}(0,c)\neq 0$ if and only if $c=\xi$, and where $\tilde{N}\left(\frac{1}{\rho},\infty\right)=0$. We next write $\tilde{\tilde{n}}(r,c)$ for the number of distinct c-points of $f(\lambda)$ in $\overline{\mathfrak{D}}_r$, and set

$$\tilde{\tilde{N}}\left(\frac{1}{\rho},c\right)=\int_0^{\frac{1}{\rho}}\frac{\tilde{\tilde{n}}(r,c)-\tilde{\tilde{n}}(0,c)}{r}dr+\tilde{\tilde{n}}(0,c)\log\frac{1}{\rho}$$

where

$$\tilde{\tilde{n}}(0,c)=\begin{cases}1 & (c=\xi)\\ 0 & (c\neq\xi)\end{cases}\qquad \tilde{\tilde{N}}\left(\frac{1}{\rho},\infty\right)=0,$$

and

$$\tilde{m}\left(\frac{1}{\rho},c\right)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}\frac{1}{\left|f\left(\frac{1}{\rho}e^{it}\right)-c\right|}dt & (c\neq\infty)\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\overset{+}{\log}\left|f\left(\frac{1}{\rho}e^{it}\right)dt\right| & (c=\infty)\end{cases}$$

for any ρ with $\sigma<\rho<+\infty$. Then if we define $\varepsilon_\kappa\left(\frac{1}{\rho},c\right)(\kappa=1,2)$ as

$$\tilde{m}\left(\frac{1}{\rho},\infty\right)-\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\left|f\left(\frac{1}{\rho}e^{it}\right)-c\right|dt=\begin{cases}\varepsilon_1\left(\frac{1}{\rho},c\right) & (c=\xi)\\ \varepsilon_2\left(\frac{1}{\rho},c\right) & (c\neq\xi,\infty),\end{cases}$$

we can find from the inequality $\overset{+}{\log}\left|\sum_{\nu=1}^{p}\alpha_\nu\right|\leqq\sum_{\nu=1}^{p}\overset{+}{\log}|\alpha_\nu|+\log p$ valid for any complex numbers α_ν that $\left|\varepsilon_\kappa\left(\frac{1}{\rho},c\right)\right|\leqq\overset{+}{\log}|c|+\overset{+}{\log}2$ for $\kappa=1,2$, and hence can analyze Nevanlinna's first fundamental theorem [1], as follows:

(B) $$\tilde{m}\left(\frac{1}{\rho},\infty\right)=\tilde{m}\left(\frac{1}{\rho},c\right)+\tilde{N}\left(\frac{1}{\rho},c\right)+K\left(\frac{1}{\rho},c\right)\qquad(\sigma<\rho<+\infty),$$

where

(C) $$K\left(\frac{1}{\rho},c\right)=\begin{cases}0 \ (\text{if } c=\infty)\\ \log|C_{-k}|+\varepsilon_1\left(\frac{1}{\rho},c\right)\ \begin{pmatrix}\text{if } c=\xi,\ C_{-\mu}=0 \text{ for}\\ 1\leqq\mu\leqq k-1, \text{ and } C_{-k}\neq0\end{pmatrix}\\ \log|\xi-c|+\varepsilon_2\left(\frac{1}{\rho},c\right)\ \ (\text{if } c\neq\xi,\infty).\end{cases}$$

In fact, for the special case $c=\xi$ we attain to the second result of (C) by considering the function

$$F(\lambda)\equiv\begin{cases}\dfrac{f(\lambda)-\xi}{(\rho\lambda)^k}=\dfrac{1}{\rho^k}\sum_{\mu=k}^{\infty}C_{-\mu}\lambda^{\mu-k} & \left(C_{-k}\neq0,\ 0<|\lambda|\leqq\dfrac{1}{\rho}\right)\\ \dfrac{C_{-k}}{\rho^k}\quad(\lambda=0)\end{cases}$$

and the other two cases are trivial.

On the assumption that

(D) $$\overline{\lim_{\rho\to\sigma+0}}\frac{\tilde{m}\left(\frac{1}{\rho},\infty\right)}{\log\{1/(\rho-\sigma)\}}=+\infty,$$

we set

$$\tilde{\delta}(c)=\underline{\lim_{\rho\to\sigma+0}}\frac{\tilde{m}\left(\frac{1}{\rho},c\right)}{\tilde{m}\left(\frac{1}{\rho},\infty\right)}=1-\overline{\lim_{\rho\to\sigma+0}}\frac{\tilde{N}\left(\frac{1}{\rho},c\right)}{\tilde{m}\left(\frac{1}{\rho},\infty\right)},$$

$$\tilde{\Theta}(c)=1-\overline{\lim_{\rho\to\sigma+0}}\frac{\tilde{\tilde{N}}\left(\frac{1}{\rho},c\right)}{\tilde{m}\left(\frac{1}{\rho},\infty\right)},$$

and $$\tilde{\theta}(c)=\underline{\lim_{\rho\to\sigma+0}}\frac{\tilde{N}\left(\frac{1}{\rho},c\right)-\tilde{\tilde{N}}\left(\frac{1}{\rho},c\right)}{\tilde{m}\left(\frac{1}{\rho},\infty\right)}.$$

Then it follows from Nevanlinna's theorem on deficient values [2] that the set of values c for which $\tilde{\Theta}(c)>0$ is countable and that

$$\sum_c \{\tilde{\delta}(c)+\tilde{\theta}(c)\} \leqq \sum_c \tilde{\Theta}(c) \leqq 2$$

where $\sum_c$ denotes the sum taken over all such values c as $\tilde{\Theta}(c)>0$. On the other hand, it is at once obvious that equality $\tilde{m}\left(\frac{1}{\rho}, c\right)= m(\rho, c)$ holds for any value c, finite or infinite. Moreover there is no difficulty in showing by direct computations that both $\tilde{N}\left(\frac{1}{\rho}, c\right)= N(\rho, c)$ and $\tilde{\bar{N}}\left(\frac{1}{\rho}, c\right)=\bar{N}(\rho, c)$ hold for every finite c and it is clear that those four quantities equal 0 for $c=\infty$. These results imply that (D) is equivalent to (A) and that the relations $\tilde{\delta}(c)=\delta(c)$, $\tilde{\Theta}(c)=\Theta(c)$, and $\tilde{\theta}(c)=\theta(c)$ hold good for every complex value c, finite or infinite.

The theorem has thus been proved.

Corollary 7. Let $U(\lambda)$ and σ be the same notations as those in Theorem 62, and let the condition (A) stated before be satisfied. Then, in the annular demain $D_\varepsilon\{\lambda: \sigma<|\lambda|<\sigma+\varepsilon\}$ for any small positive ε, $U(\lambda)$ assumes infinitely often all finite complex values except at most one (finite value). In addition, if c is a non-exceptional finite value of $U(\lambda)$ for D_ε and if $\alpha_{p_1}^{(c)}, \alpha_{p_2}^{(c)}, \alpha_{p_3}^{(c)}, \cdots$ denote distinct c-points of $U(\lambda)$ in D_ε, the positive series $\sum_{\nu=1}^{\infty}(|\alpha_{p_\nu}^{(c)}|-\sigma)$ is divergent.

Proof. Since, by hypotheses, $U(\lambda)$ is regular in the domain $\{\lambda: \sigma<|\lambda|\leqq+\infty\}$ and assumes the value ξ at $\lambda=\infty$, and since Theorem 62 implies that the number of exceptional values of $U(\lambda)$ for D_ε never exceeds 2, the former half of the corollary is a direct consequence of this theorem. Furthermore the latter half is deduced immediately from the fact that, according to Theorem 62,

$$\bar{N}(\rho, c)=\log\frac{|\alpha_1^{(c)}\alpha_2^{(c)}\cdots\alpha_{\bar{n}(\rho,c)-\bar{n}(\infty,c)}^{(c)}|}{\rho^{\bar{n}(\rho,c)-\bar{n}(\infty,c)}}-\bar{n}(\infty, c)\log\rho$$

$$(\bar{n}(\infty, \xi)=1,\ \bar{n}(\infty, c)=0 \text{ for } c\neq\xi)$$

$$=\log\prod_{\mu=1}^{\bar{n}(\rho,c)-\bar{n}(\infty,c)}\left(1+\frac{|\alpha_\mu^{(c)}|-\rho}{\rho}\right)-\bar{n}(\infty, c)\log\rho\to+\infty(\rho\to\sigma+0)$$

where $\alpha_\mu^{(c)}(\mu=1, 2, 3, \cdots, \bar{n}(\rho, c)-\bar{n}(\infty, c))$ are distinct c-points of $U(\lambda)$ in the domain $\{\lambda: \rho\leqq|\lambda|<+\infty\}$ with $\sigma<\rho<+\infty$ and that the number of distinct c-points of $U(\lambda)$ in the domain $\{\lambda: \sigma+\varepsilon\leqq|\lambda|<+\infty\}$ is at most finite.

Theorem 63. Let the ordinary part of $U(\lambda)$ be a polynomial in λ. Then the results of Theorem 62 are also valid for this $U(\lambda)$.

Proof. Suppose that the ordinary part of $U(\lambda)$ is given by $\sum_{\mu=0}^{d} e_\mu \lambda^\mu (e_d \neq 0)$. For case $c \neq \infty$ we consider the function $\varphi(\lambda, c)$ defined by

$$\varphi(\lambda, c)=\begin{cases}(\rho\lambda)^d\left[U\left(\frac{1}{\lambda}\right)-c\right] & \left(0<|\lambda|\leqq\frac{1}{\rho}, \sigma<\rho+<\infty\right)\\ e_d\rho^d \quad (\lambda=0).\end{cases}$$

Evidently $\varphi(\lambda,c)$ is regular in the closed domain $\overline{\mathfrak{D}}_{\rho^{-1}}\left\{\lambda\colon 0\leqq|\lambda|\leqq\frac{1}{\rho}\right\}$. We now write $n'(r, c)$ for the number of zeros, with due count of multiplicity, of $\varphi(\lambda, c)$ in $\overline{\mathfrak{D}}_r$ with $r\leqq\frac{1}{\rho}$ and $\bar{n}'(r, c)$ for the number of distinct zeros of $\varphi(\lambda, c)$ in $\overline{\mathfrak{D}}_r$. By using $n'(r, c)$ and $\bar{n}'(r, c)$ we define $N'\left(\frac{1}{\rho}, c\right)$ and $\bar{N}'\left(\frac{1}{\rho}, c\right)$ respectively, as $\tilde{N}\left(\frac{1}{\rho}, c\right)$ and $\bar{\tilde{N}}\left(\frac{1}{\rho}, c\right)$ were introduced from $\tilde{n}(r, c)$ and $\bar{\tilde{n}}\ (r, c)$ respectively; and we write

$$m'\left(\frac{1}{\rho}, c\right)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\frac{1}{\left|\varphi\left(\frac{1}{\rho}e^{it}, c\right)\right|}dt \qquad (\sigma<\rho<+\infty),$$

$$m'\left(\frac{1}{\rho}, \varphi_c\right)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}\left|\varphi\left(\frac{1}{\rho}e^{it}, c\right)\right|dt \qquad (\sigma<\rho<+\infty).$$

On setting $\varepsilon\left(\frac{1}{\rho}, c\right)=m'\left(\frac{1}{\rho}, \varphi_0\right)-m'\left(\frac{1}{\rho}, \varphi_c\right)$, it is verified in the same manner as before that

$$\text{(E)} \qquad m'\left(\frac{1}{\rho}, \varphi_0\right)=m'\left(\frac{1}{\rho}, c\right)+N'\left(\frac{1}{\rho}, c\right)$$
$$+\log|e_d|+[d+n'(0, c)]\log\rho+\varepsilon\left(\frac{1}{\rho}, c\right),$$

where $\varepsilon\left(\frac{1}{\rho}, 0\right)=0$. On the other hand,

$$m'\left(\frac{1}{\rho}, \varphi_c\right)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|U(\rho e^{-it})-c|\,dt;$$

and if we denote the right-hand member of this equality by $m(\rho, U-c)$ as usual, then

$$|m(\rho, U-c)-m(\rho, U)|\leqq\overset{+}{\log}|c|+\log 2$$

and therefore $\left|\varepsilon\left(\frac{1}{\rho}, c\right)\right|\leqq\overset{+}{\log}|c|+\log 2$ for $c\neq 0$. For the special case $c=\infty$, the quantities $N'\left(\frac{1}{\rho}, c\right)$, $\bar{N}'\left(\frac{1}{\rho}, c\right)$, and $m'\left(\frac{1}{\rho}, c\right)$ should be defined directly from the function $f(\lambda)=U\left(\frac{1}{\lambda}\right)$ with domain $\overline{\mathfrak{D}}_{\rho^{-1}}$ and so it turns out as before that the equalities $N'\left(\frac{1}{\rho}, \infty\right)=$

$N(\rho, \infty)=-d\log\rho$, $\bar{N}'\left(\frac{1}{\rho}, \infty\right)=\bar{N}(\rho, \infty)=-\log\rho$, and $m'\left(\frac{1}{\rho}, \infty\right)=m(\rho, \infty)=\frac{1}{2\pi}\int_0^{2\pi}\overset{+}{\log}|U(\rho e^{-it})|\,dt$ hold for the notations $N(\rho, \infty)$, $\bar{N}(\rho, \infty)$, and $m(\rho, \infty)$ associated with our $U(\lambda)$ by the same methods as in Theorem 62. Hence we have

$$m'\left(\frac{1}{\rho}, \varphi_0\right)=m'\left(\frac{1}{\rho}, \infty\right), \tag{F}$$

and the equalities (E) and (F) lead us to the result that

$$m'\left(\frac{1}{\rho}, \infty\right)=m'\left(\frac{1}{\rho}, c\right)+N'\left(\frac{1}{\rho}, c\right)+K'\left(\frac{1}{\rho}, c\right),$$

where $$K'\left(\frac{1}{\rho}, c\right)=\begin{cases}\log|e_d|+[d+n'(0, c)]\log\rho+\varepsilon\left(\frac{1}{\rho}, c\right) & (c\neq\infty)\\ d\log\rho & (c=\infty).\end{cases}$$

This result corresponds to (B) in the course of the proof of Theorem 62. Since, moreover, the equalities $N'\left(\frac{1}{\rho}, c\right)=N(\rho, c)$, $\bar{N}'\left(\frac{1}{\rho}, c\right)=\bar{N}(\rho, c)$, and $m'\left(\frac{1}{\rho}, c\right)=m(\rho, c)$ hold for every complex number c, finite or infinite, the validity of the present theorem is shown by the same reasoning as that used to prove the preceding one.

Corollary 8. The results of Corollary 7 hold also for the function $U(\lambda)$ in Theorem 63.

Proof. This corollary is a direct consequence of Theorem 63, as will be found immediately from the reasoning used to prove Corollary 7.

Remark. Since, as demonstrated above, Nevanlinna's first fundamental theorem and Nevanlinna's theorem on deficient values both are extended in the domain $\{\lambda: \sigma<|\lambda|\leqq+\infty\}$ to the functions $U(\lambda)$ of Theorems 62 and 63 provided that these functions satisfy (A), so also is the Nevanlinna theory itself based on his two theorems.

Correction to "Some applications of the functional-representations of normal operators in Hilbert spaces. XXI" (Proc. Japan Acad., 42, pp. 583–588 (1966)).

Page 587 line 6 should read "$=\left|\sum_{\alpha=1}^{\infty}\int_{\{\lambda_\nu\}_{\nu\geqq p}}\cdots\right|$", instead of "$=\left|\sum_{\alpha=1}^{\infty}\int_{\{\lambda_\nu\}_{\nu\leqq p}}\cdots\right|$".

References

[1] R. Nevanlinna: Eindeutige analytische Funktionen. Springer, Berlin (1936).
[2] W. K. Hayman: Meromorphic Functions. Oxford (1964).