163. *Boundary Value Problems for the Helmholtz Equations. II*

The Case of Parallel Lines with Openings

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(Comm. by Kinjirô KUNUGI, M.J.A., Sept. 12, 1966)

1. In the preceding paper [1], the author has solved two kinds of boundary value problems for the Helmholtz equations in domains bounded by coaxial circles with arbitrary number of arbitrary openings in them. These correspond to the E and H electromagnetic fields in media, which are not necessarily the same, occupying contiguous coaxial circular domains separated by circular boundaries of perfect conductivity which having arbitrary slots in them. In this paper, the same approach is applied to solve the Helmholtz equations in domains separated by parallel lines with arbitrary openings in them. This corresponds to the E and H waves in media, which are not necessarily identical, in domains separated by parallel gratings of plane strips of arbitrary width.

The simplest problem of diffraction of electromagnetic waves by a grating is that when the grating is composed of (i) a single, (ii) infinitely long series of (iii) equally spaced obstacles of a regular geometry, (iv) in a uniform medium and when (v) a plane wave is incident on it. If we replace one or more of these conditions (i)-(v) by some other conditions, then the problem will be generalized in various ways. For example, (ii) has been generalized to a case of a grating of finite or semi-infinite series of an identical cylinder [2].

In this paper, the conditions (i), (iii), and (iv) are generalized and it is assumed that there are two parallel gratings of plane strips, composed of arbitrary number of line segments of arbitrary length, and that the media separated by these gratings are not necessarily identical. Note that the method is generalized, in a way similar to that mentioned in the previous paper [1], to the case where the number of gratings is more than two. In this paper, it is also assumed that the gratings are of periodic structure in the large, though they may be non-periodic locally. The result is expected to be generalized to the case where there is no such periodicity.

2. Let b and d be positive real numbers, and that L_j ($j=1, 2$)

be unions of ν_j line segments in the xy-pl., respectively, such that

L_1; $x=0$, $-b\leqq y_{1,1}<y<y_{1,2}$, $y_{1,3}<y<y_{1,4}$, $\cdots$, $y_{1,2\nu_1-1}<y<y_{1,2\nu_1}\leqq b$,

L_2; $x=d$, $-b\leqq y_{2,1}<y<y_{2,2}$, $y_{2,3}<y<y_{2,4}$, $\cdots$, $y_{2,2\nu_2-1}<y<y_{2,2\nu_2}\leqq b$,

where $\{y_{1,m}\}$ and $\{y_{2,m}\}$ are given sets of increasing real numbers. Suppose that L_1^c and L_2^c are unions of open intervals whose closure being the complements of L_1 and L_2 with respect to the line segments $x=0$, $-b\leqq y\leqq b$, and $x=d$, $-b\leqq y\leqq b$, respectively. Assume that the gratings are composed of L_1 and L_2 and their periodical repetition with the period $2b$ from $y=-\infty$ to $y=\infty$. Finally, let the domains separated by these gratings be denoted by S_j $(j=1, 2, 3)$, that is, S_1; $x<0$, S_2; $0<x<d$, S_3; $d<x$, and let the medium constants of the media occupying S_j be k_j. Note that the structure of these gratings is periodic in the large with the period $2b$, but is arbitrary in each interval $-b\leqq y\leqq b$. Then, our problem is stated by (1), (2), (3), (4), or (4)′ of the previous paper [1], if $\nu=2$ and $f_j=0$ in the right hand side of (1), if L_j are understood not to be those defined in [1] but to be these unions of line segments defined above, and if the incident plane wave $u^{(i)}=fe^{-ik_1x\cos\theta_0-ik_1y\sin\theta_0}$ is assumed in S_1, where f and θ_0 are given constants and are the amplitude (including zero) and the angle of incidence of $u^{(i)}$, respectively.

To begin with, as the solutions of (1) of [1] satisfying (3) of [1], solutions of the Helmholtz equations in S_j are necessarily expressed by

$$
\begin{aligned}
(1)\qquad & u_1(x, y)=\sum A_n e^{ik_{1,n}x+i(\beta n-k_1\sin\theta_0)y}+u^{(i)}, && \text{in } S_1,\\
& u_2(x, y)=\sum \{B_n e^{ik_{2,n}x}+C_n e^{-ik_{2,n}x}\}e^{i(\beta n-k_1\sin\theta_0)y}, && \text{in } S_2,\\
& u_3(x, y)=\sum D_n e^{-ik_{3,n}x+i(\beta n-k_1\sin\theta_0)y}, && \text{in } S_3,
\end{aligned}
$$

where A_n, B_n, C_n, and D_n are undetermined constants and

$$(2)\qquad B=\pi/b,\quad k_{jn}^2=k_j^2-(\beta n-k_1\sin\theta_0)^2,\quad \text{Im. } k_{jn}\leqq 0,\qquad (j=1, 2, 3).$$

As was mentioned in [1], our problem is equivalent to determine A's, B's, C's, and D's so that u's satisfy the following sets of conditions;

Problem E.

$$(3)\qquad u_j=u_{j+1}\qquad \text{on } L_j+L_j^c,$$

$$(4)\qquad u_2=0,\qquad \text{on } L_1 \text{ and } L_2,$$

$$(5)\qquad \frac{\eta_j}{k_j}\frac{\partial u_j}{\partial x}-\frac{\eta_{j+1}}{k_{j+1}}\frac{\partial u_{j+1}}{\partial x}=\begin{cases}0, & \text{on } L_j^c,\\ 2ibe^{-ik_1y\sin\theta_0}\tau_j(y), & \text{on } L_j,\end{cases}\qquad (j=1, 2)$$

and

Problem H.

$$(6)\qquad \frac{\eta_j}{k_j}\frac{\partial u_j}{\partial x}=\frac{\eta_{j+1}}{k_{j+1}}\frac{\partial u_{j+1}}{\partial x},\qquad \text{on } L_j+L_j^c,$$

$$
\begin{aligned}
(7)\qquad & \frac{\partial u_1}{\partial x}=\begin{cases}0, & \text{on } L_1,\\ 2ibe^{-ik_1y\sin\theta_0}\tau_1(y) & \text{on } L_1^c,\end{cases}\\
& \frac{\partial u_3}{\partial x}=\begin{cases}0, & \text{on } L_2,\\ -2ibe^{-ik_1y\sin\theta_0}\tau_2(y), & \text{on } L_2^c,\end{cases}
\end{aligned}
$$

(8) $$u_j = u_{j+1} \quad \text{on } L_j^c, \qquad (j=1, 2)$$

where η_j are given constants and τ_j are unknown functions defined on L_j and L_j^c by the left hand members of (5) and (7), respectively. Additional conditions on these problems are the edge conditions, which are stated as follows; Let l_j represent L_j for Problem E and L_j^c for Problem H. Suppose that y is the y-coordinate of a point on l_j, and that y_{jm} are the y-coordinates of end points of the line segments composing l_j. Then, it is required that $\tau_j(y)$ are of the form

(9) $$\tau_j(y) = \frac{\tau_j^*(y)}{\sqrt{y - y_{jm}}}$$

when the point y is on l_j and in the vicinity of an end point y_{jm}, where $\tau_j^*(y)$ are Hölder continuous at any point on l_j including the end points of l_j.

3. First, we will study Problem E. On substituting (1) into (3) and (5) and on making use of the orthogonality of $\{e^{i\beta ny}\}$ over $L_j + L_j^c$, we are left with simultaneous linear equations with respect to A_n, B_n, C_n, and D_n which are solved to give

(10) $$\begin{aligned} \Delta_n A_n &= \{\lambda_{2n} \cos k_{2n} d + i\lambda_{3n} \sin k_{2n} d\} \mathfrak{A}_{1n} + \lambda_{2n} \mathfrak{A}_{2n} + g_{1n}, \\ \Delta_n B_n &= \tfrac{1}{2} e^{-ik_{2n}d} (\lambda_{2n} - \lambda_{3n})(\mathfrak{A}_{1n} + f_n) + \tfrac{1}{2}(\lambda_{1n} + \lambda_{2n}) \mathfrak{A}_{2n}, \\ \Delta_n C_n &= \tfrac{1}{2} e^{ik_{2n}d} (\lambda_{2n} + \lambda_{3n})(\mathfrak{A}_{1n} + f_n) - \tfrac{1}{2}(\lambda_{1n} - \lambda_{2n}) \mathfrak{A}_{2n}, \\ \Delta_n D_n &= e^{ik_{3n}d} \{\lambda_{2n}(\mathfrak{A}_{1n} + f_n) + \mathfrak{A}_{2n}(\lambda_{2n} \cos k_{2n} d + i\lambda_{1n} \sin k_{2n} d)\}, \end{aligned}$$

where g_{1n} are certain given constants, $f_n = 2f\lambda_{1n}\delta_{n,0}$, and

(11) $$\lambda_{jn} = \frac{\eta_j}{k_j} k_{jn}, \qquad \mathfrak{A}_{jn} = \int_{L_j} \tau_j(y) e^{-i\beta ny} dy,$$
$$\Delta_n = \lambda_{2n}(\lambda_{1n} + \lambda_{3n}) \cos k_{2n} d + i(\lambda_{1n}\lambda_{3n} + \lambda_{2n}^2) \sin k_{2n} d.$$

On substituting (10) into (4), we have simultaneous integral equations with respect to the unknown functions $\tau_j(y)$, which are written, for $j=1$ and 2, as

(12) $$\int_{L_j} \tau_j(\eta) \sum S_n^{jj} e^{i\beta n(y-\eta)} d\eta + \int_{L_k} \tau_k(\eta) \sum S_n^{jk} e^{i\beta n(y-\eta)} d\eta = F_j,$$

where F_j are given constants, and $j, k = 1, 2$ and $j \neq k$. In (12), S's are given by

(13) $$\Delta_n S_n^{1,1} = \lambda_{2n} \cos k_{2n} d + i\lambda_{3n} \sin k_{2n} d, \qquad \Delta_n S_n^{1,2} = \Delta_n S_n^{2\,1} = \lambda_{2n},$$
$$\Delta_n S_n^{2,2} = i\lambda_{1n} \sin k_{2n} d + \lambda_{2n} \cos k_{2n} d.$$

If we find solutions τ_j of eq.s (12) which satisfy the conditions (9), then, successively, $\mathfrak{A}_{jn}$ $(j=1, 2)$ are obtained by (11), A_n, B_n, C_n, and D_n are obtained by (10) and finally u_j are obtained by (1). Furthermore, it is proved that the u's thus obtained satisfy all requirements of Problem E, including the edge conditions (9). Hence, the eq.s (12) are fundamental to Problem E.

In a way similar to this, it is proved that the eq.s (12) are the fundamental equations of Problem H, if L_j and L_k are replaced by L_j^c and L_k^c respectively, and if S_n^{jj} and S_n^{jk} are understood not to be those defined by (13) but to be those defined appropriately for Problem H, though the details are not described here. In fact, on substituting (1) into (6) and (7), we have simultaneous linear equations, which are solved for A_n, B_n, C_n, and D_n in terms of $\mathfrak{A}_{jn}$ which being the integrals of τ_j over L_j^c. Then, with help of these results, (8) is reduced to (12).

The eq.s (12) are the first kind integral equations of Fredholm, whose kernels having a log singularity as shown below. Thus, our two problems E and H have been reduced to that solving for the fundamental integral equations, which being formally the same for both of the problems, whose solutions solving both the problems E and H simultaneously.

4. If we put $k_{jn}=-i\beta|n|\{1+\delta_n\}$ for $n\neq 0$, then, δ_n is as small as any given positive number if $|n|$ is so large. Accordingly, we may see that if

$$S_n^{jj}=\frac{c_{jj}}{|n|}\{1+s_n^{jj}\}, \qquad S_n^{jk}=\frac{c_{jk}}{|n|}e^{-ik_{2n}d}\{1+s_n^{jk}\}, \qquad (j\neq k),$$

where c_{jj} and c_{jk} are certain known constants independent of n, then, s_n^{jj} and s_n^{jk} are quantities of order $1/n$ if $n>N$ where N is a sufficiently large positive integer. Hence, the kernels of eq.s (12) are approximated by;

$$\sum_{n=-\infty}^{\infty} S_n^{jj}e^{i\beta n(y-\eta)}=c_{jj}\log. 1/\{2-2\cos\beta(y-\eta)\}+S_0^{jj}+\sum_{\substack{n=-N\\ n\neq 0}}^{N}\frac{c_{jj}}{|n|}s_n^{jj}e^{i\beta n(y-\eta)}$$

and

$$\sum_{n=-\infty}^{\infty} S_n^{jk}e^{i\beta n(y-\eta)}=\sum_{n=-N}^{N} S_n^{jk}e^{i\beta n(y-\eta)},$$

respectively. Consequently, the theory developed in [3] is applicable to the eq.s (12). On solving for the eq.s (12), it is shown, after some calculations, that the solutions (1) are given by

$$(14) \qquad u_j(x, y)=\sum e^{i(\beta n-k_1\sin\theta_0)y}\cdot U_{jn}(x), \qquad (j=1, 2, 3)$$

where, if $\varDelta_n\neq 0$ for any n,

$$(15) \quad U_{1n}(x)=\frac{1}{\varDelta_n}e^{ik_{1n}x}\{(\lambda_{2n}\cos k_{2n}d+i\lambda_{3n}\sin k_{2n}d)\mathfrak{A}_{1n}+\lambda_{2n}\mathfrak{A}_{2n}\}$$
$$+\frac{1}{\varDelta_n}f\delta_{n,0}(h^+e^{ik_{1n}x}+h^-e^{-ik_{1n}x})$$

$$U_{2n}(x)=\frac{1}{\Delta_n}\{\lambda_{2n}\cos k_{2n}(x-d)-i\lambda_{3n}\sin k_{2n}(x-d)\}(\mathfrak{A}_{1n}+f_n)$$
$$+\frac{1}{\Delta_n}\{\lambda_{2n}\cos k_{2n}x+i\lambda_{1n}\sin k_{2n}x\}\mathfrak{A}_{2n},$$
$$U_{3n}(x)=\frac{1}{\Delta_n}e^{-ik_{3n}(x-d)}\{\lambda_{2n}(\mathfrak{A}_{1n}+f_n)+(\lambda_{2n}\cos k_{2n}d+i\lambda_{1n}\sin k_{2n}d)\mathfrak{A}_{2n}\}.$$

where $h_n^{\pm}$ are certain given constants, for Problem E, and

$$(16)\quad U_{1n}(x)=\frac{e^{ik_{1n}x}}{k_{1n}}\mathfrak{A}_{1n}+2f\delta_{n,0}\cos k_{1n}x,$$
$$U_{2n}(x)=\frac{-1}{i\lambda_{2n}\sin k_{2n}d}\left\{\frac{\eta_1}{k_1}\mathfrak{A}_{1n}\cos k_{2n}(x-d)+\frac{\eta_3}{k_3}\mathfrak{A}_{2n}\cos k_{2n}x\right\},$$
$$U_{3n}(x)=\frac{e^{-ik_{3n}(x-d)}}{k_{3n}}\mathfrak{A}_{2n},$$

for Problem H. Furthermore, it is shown [3] that

$$(17)\qquad \mathfrak{A}_{jn}=\frac{1}{i\beta}\sum_{m=-N}^{N+\nu_j}p_m^j\alpha_{m-n}^j \qquad (j=1,2)$$

where

$$\alpha_n^j=\frac{1}{b}\int_{l_j}X_j(y)e^{i\beta ny}dy,\qquad X_j(y)=\left\{\prod_{m=1}^{2\nu_j}(e^{i\beta y}-e^{i\beta y_{jm}})\right\}^{-\frac{1}{2}},$$

and $\{p_m^j\}$ are constants determined by certain simultaneous linear equations.

(14), (15), (16), and (17) are the complete explicit formulations for the solutions of Problems E and H, which satisfy all requirements of the problems including the edge conditions (9), and which are true everywhere for any wave number. The terms of $\mathfrak{A}_{jn}$ in the right hand members of (15) and (16) represent the effects of the gratings L_j.

We have to have further discussions on, for example, the resonance cases where $\Delta_n=0$, $k_{1n}=0$, etc., in (15) and (16). It is shown, for example, that there occur seven cases in which $\Delta_n=0$, in each of which the Wood's anomalies are studied. However, these discussions and detailed descriptions of expressions which have not been given above are expected to be published in their full text in some journal soon.

References

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