162. Boundary Value Problems for the Helmholtz Equations. I

The Case of Coaxial Circular Arcs

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1. Let (r, θ) be polar coordinates in a plane and let S_j be domains defined by $S_1; r<a_1$, $S_j; a_{j-1}<r<a_j$, $(j=2, 3, \cdots, \nu)$, $S_{\nu+1}$; $a_\nu<r$. $(a_1<a_2<\cdots<a_\nu)$. Suppose that, for each $j=1, 2, \cdots, \nu$, L_j is a union of arbitrary (but finite) number of circular arcs of arbitrary width and of radius a_j, and that L_j^c is the complement of L_j with respect to the whole circle $r=a_j$. Then, our problems are stated as follows; Find functions $u_j(r, \theta)$ in S_j such that their partial derivatives of the second order are continuous in S_j excepting given points $x_j^* \in S_j$, that u_j and $\partial u_j/\partial r$ are Hölder continuous in the closure of S_j, and that they satisfy

(1) $\Delta u_j + k_j^2 u_j = f_j \delta(x, x_j^*), \quad x \in S_j, \ x_j^* \in S_j, \qquad (j=1, 2, \cdots, \nu+1)$

(2) u and $\dfrac{\eta}{k}\dfrac{\partial u}{\partial r}$ are continuous when they traverse L_j^c.

(3) $$\lim \cdot r^{\frac{1}{2}}\left\{\frac{\partial u_{\nu+1}}{\partial r} + ik_{\nu+1}u_{\nu+1}\right\}=0, \qquad r\longrightarrow\infty,$$

and

(4) $u_j=0$ on $L_{j-1}+L_j$, $(j=2, 3, \cdots, \nu)$, $u_1=0$ on L_1 and $u_{\nu+1}=0$ on L_ν,

or

(4)′ $$\frac{\partial u_j}{\partial r}=0 \text{ on } L_{j-1}+L_j, \qquad (j=2, 3, \cdots, \nu),$$

$$\frac{\partial u_1}{\partial r}=0 \text{ on } L_1 \text{ and } \frac{\partial u_{\nu+1}}{\partial r}=0 \text{ on } L_\nu,$$

where Δ is the two-dimensional Laplace operator and k_j $(j=1, 2, \cdots, \nu+1)$ are complex constants where $\mathrm{Im}\cdot k_j \leqq 0$. $\eta=\eta_j$ and f_j are given complex constants where f_j may include zero but $\eta_j \neq 0$. These are the simultaneous boundary value problems for the Helmholtz equations in contiguous domains bounded by circular arcs, in which the parameters k_j are not necessarily uniform. They are a generalization of the boundary value problem for the single Helmholtz equation, and is also a generalization of the theory of electromagnetic fields in a uniform medium bounded by circular arcs

[1]. In the following, it will be shown how the problems are solved generally by means of simultaneous integral equations of Fredholm of the first kind, and then, detailed discussions will be given in the case of two ($\nu=2$) coaxial circular arcs, obtaining explicit formulations of the solutions.

2. As the solution of (1), u_j are necessarily expressed by

(5) $$u_j(r,\theta)=\sum_n \{\alpha_n^j J_n(k_j r)+\beta_n^j H_n(k_j r)\}e^{in\theta}+u_j^*(r,\theta),$$

where α_n^j and β_n^j are unknown constants, J_n is the Bessel function, H_n is the Hankel function of the second kind, and $u_j^*(r,\theta)=f_j H_0(k_j R_j)$ where R_j is the distance from the source point x_j^*. Because of the boundedness of $u_1(r,\theta)$ at $r=0$ and (3), it is understood that $\beta_n^1=\alpha_n^{\nu+1}=0$ in (5).

Now, it is easy to see that the conditions (2) and (4) are equivalent to the following conditions (6), (7), and (8), while (2) and (4)′ are equivalent to (9), (10), and (11);

Problem E.

(6) $$u_j=u_{j+1} \quad \text{on } L_j+L_j^c,$$

(7) $$u_j=0 \quad \text{on } L_j,$$

(8) $$\frac{\eta_j}{k_j}\frac{\partial u_j}{\partial r}-\frac{\eta_{j+1}}{k_{j+1}}\frac{\partial u_{j+1}}{\partial r}=\begin{cases}0 & \text{on } L_j^c,\\ 2\pi\tau_j & \text{on } L_j,\end{cases} \qquad (j=1,2,\cdots,\nu)$$

Problem H.

(9) $$\frac{\eta_j}{k_j}\frac{\partial u_j}{\partial r}=\frac{\eta_{j+1}}{k_{j+1}}\frac{\partial u_{j+1}}{\partial r} \quad \text{on } L_j+L_j^c,$$

(10) $$\frac{\partial u_j}{\partial r}=\begin{cases}0, & \text{on } L_j,\\ 2\pi\tau_j, & \text{on } L_j^c,\end{cases}$$

(11) $$u_j=u_{j+1} \quad \text{on } L_j^c, \qquad (j=1,2,\cdots,\nu)$$

where τ_j are unknown functions defined on L_j and L_j^c by the left hand members of (8) and (10), respectively. We will impose additional conditions on the problems, that is, the edge conditions at the end points of arcs composing L_j. These are stated as follows; Let l_j be L_j in the case of (8) and be L_j^c in the case of (10), and let (a_j,θ_{jm}) $(m=1,2,\cdots,2\nu_j)$ be end points of ν_j arcs composing l_j. Then, the conditions say that τ_j at a point (a_j,θ) on l_j near an end point (a_j,θ_{jm}) is of the form

(12) $$\tau_j(\theta)=\frac{\tau_j^*(\theta)}{\sqrt{e^{i\theta}-e^{i\theta_{jm}}}}$$

where $\tau_j^*(\theta)$ is Hölder continuous on l_j including the end point (a_j,θ_{jm}).

To begin with, the problem E will be studied. On substituting (5) into (6) and making use of the orthogonality of $\{e^{in\theta}\}$ over the whole circle $L_j+L_j^c$, we have

(13) $J_n(k_ja_j)\alpha_n^j + H_n(k_ja_j)\beta_n^j - J_n(k_{j+1}a_j)\alpha_n^{j+1} - H_n(k_{j+1}a_j)\beta_n^{j+1} = v_{jn}.$

Similarly, on substituting (5) into (8), we have

(14) $\eta_j J_n'(k_ja_j)\alpha_n^j + \eta_j H_n'(k_ja_j)\beta_n^j - \eta_{j+1}J_n'(k_{j+1}a_j)\alpha_n^{j+1} - \eta_{j+1}H_n'(k_{j+1}a_j)\beta_n^{j+1} = \mathfrak{A}_{jn} + w_{jn},$

where v_{jn} and w_{jn} are known quantities composed of $u_j^*(a_j, \theta)$ whose detailed expressions however, for the sake of simplicity, are not described here. Prime ($'$) denotes the differentiation with respect to the argument and $\mathfrak{A}_{jn}$ are constants defined by

$$\mathfrak{A}_{jn} = \int_{L_j} \tau_j(\phi) e^{-in\phi} d\phi. \tag{15}$$

Being 2ν linear algebraic equations with respect to 2ν unknown coefficients α_n^j and β_n^j, (13) and (14) are solved to give

$$\alpha_n^j = \sum_{k=1}^{\nu} A_n^{jk}\mathfrak{A}_{kn} + a_n^j, \tag{16}$$

$$\beta_n^j = \sum_{k=1}^{\nu} B_n^{jk}\mathfrak{A}_{kn} + b_n^j,$$

where A_n^{jk}, B_n^{jk}, a_n^j and b_n^j are known constants.

On the other hand, if we substitute (5) and (16) in (7) and if we assume that the order of the summation and the integration is interchangeable, we have

$$\sum_{k=1}^{\nu} \int_{l_k} \tau_k(\phi) \sum_n S_n^{jk} e^{in\Theta} d\phi = g_j(\theta), \qquad (j=1, 2, \cdots, \nu), \tag{17}$$

where, respectively, $l_j = L_j$, $\Theta = \theta - \phi$, $\theta \in l_j$, $g_j(\theta)$ are known functions and S_n^{jk} are known constatns defined by

$$S_n^{jk} = J_n(k_ja_j)A_n^{jk} + H_n(k_ja_j)B_n^{jk}. \tag{18}$$

Eq.s (17) are simultaneous integral equations of Fredholm of the first kind with respect to the unknowns τ_j. If the solutions τ_j of (17), which satisfy the edge conditions (12), are found, then, successively, α's and β's are obtained by (16) and u's are obtained by (5). Conversely, it is almost evident that the u's thus obtained satisfy all the requirements of Problem E. Hence, eq.s (17) are fundamental equations for the problem E.

In a way similar to this, it is shown that eq.s (17) are the fundamental ones to the problem H, where however, $\tau_j(\phi)$ should be understood to be those functions defined by (10), $l_j = L_j^c$ and S_n^{jk} and $g_j(\theta)$ to be known quantities but different from those defined in the case of Problem E, respectively. In fact, (9) and (10) are reduced to simultaneous linear equations with respect to α's and β's, which are solved to give α's and β's in terms of $\mathfrak{A}_{jn}$, where $\mathfrak{A}_{jn}$ are defined by (15) if L_j is replaced by L_j^c. Then, (11) is reduced to the eq.s (17).

As was shown above, both of Problems E and H have been reduced to the equivalent ones of solving for the fundamental equations, which are formally the same for both of them. Hence, we can solve both the problems simultaneously by solving a system of equations (17).

3. Since we have no enough space to describe the details on the general case of ν boundaries, we will confine ourselves to the case of $\nu=2$ henceforth. However, the essentials of the analysis will not be lost by this restriction, and one may easily generalize the method to the case where $\nu\geqq 3$. When $\nu=2$, that is, when we consider three domains bounded by two coaxial circles with arbitrary openings, S_n^{jk} defined by (18) for Problem E are given by

(19) $$\Delta_n S_n^{1,1}=J_n(a_1)\Gamma_n(a_2, b_2), \qquad \Delta_n S_n^{1,2}=H_n(b_3)\Lambda_n(a_2, a_2),$$
$$\Delta_n S_n^{2,1}=J_n(a_1)\Gamma_n(b_2, b_2), \qquad \Delta_n S_n^{2,2}=H_n(b_3)\Lambda_n(b_2, a_2)$$

where we have set $a_1=a, a_2=b$, and have employed abbreviated notations;

(20) $$a_j=k_j a,\ b_j=k_j b,\ r_j=k_j r, \qquad (j=1, 2, 3)$$
$$D_n(r, \rho)=J_n(r)H_n(\rho)-J_n(\rho)H_n(r),$$
$$L_n(r, \rho)=\frac{\partial D_n(r, \rho)}{\partial\rho}, \qquad L_n'(r, \rho)=\frac{\partial^2 D_n(r, \rho)}{\partial r\partial\rho}.$$
$$\Gamma_n(r, \rho_j)=\eta_3 H_n'(b_3)D_n(r, \rho_j)-\eta_j H_n(b_3)L_n(r, \rho_j),$$
$$\Gamma_n'(r, \rho_j)=\frac{\partial\Gamma_n(r, \rho_j)}{\partial\rho_j},$$
$$\Lambda_n(r, \rho_j)=\eta_1 J_n'(a_1)D_n(r, \rho_j)-\eta_j J_n(a_1)L_n(r, \rho_j).$$

(21) $$\Delta_n=\eta_1 J_n'(a_1)\Gamma_n(a_2, b_2)$$
$$+\eta_2 J_n(a_1)\{\eta_2 H_n(b_3)L_n'(a_2, b_2)+\eta_3 H_n'(b_3)L_n(b_2, a_2)\}.$$

On the other hand, S_n^{jk} for Problem H are given by

(22) $$L_n'(a_2, b_2)S_n^{1,1}=\frac{\eta_2 J_n(a_1)}{\eta_1 J_n'(a_1)}L_n'(a_2, b_2)-L_n(a_2, b_2),$$
$$L_n'(a_2, b_2)S_n^{1,2}=\frac{-2i}{\pi a_2}, \qquad L_n'(a_2, b_2)S_n^{2,1}=\frac{2i}{\pi b_2},$$
$$L_n'(a_2, b_2)S_n^{2,2}=-\frac{\eta_2 H_n(b_3)}{\eta_3 H_n'(b_3)}L_n'(b_2, a_2)+L_n(b_2, a_2).$$

The essential point of our method is to show that S_n^{jk} satisfy

(23) $$S_n^{jk}=S_{-n}^{jk},$$

(24) $$S_n^{jj}=\frac{c_{jj}}{|n|}\{1+s_{|n|}^{jj}\}, \quad S_n^{jk}=\frac{c_{jk}}{|n|}\varepsilon^n\{1+s_{|n|}^{jk}\}, \qquad (j\neq k),$$

where c_{jj} and c_{jk} are constatns independent of n, $s_{|n|}^{jj}$ and $s_{|n|}^{jk}$ are quantities of order $1/n$, and $\varepsilon<1$. In the present case, it is proved, by virtue of the formulas $J_n=(-1)^n J_{-n}$ and $H_n=(-1)^n H_{-n}$, that S's defined by (19) as well as by (22) satisfy (23), while (24) is proved with help of the author's formulas on the estimation of J_n and H_n [1]. When (23) and (24) hold, it is proved that the kernel $\sum_{n=-\infty}^{\infty} S_n^{jj} e^{in\theta} = S_0^{jj} + 2\sum_{n=1}^{\infty} S_n^{jj} \cos n\Theta$ has the log singularity if one notes that $2\sum_{n=1}^{\infty} \cos n\Theta/n = \log 1/(2-2\cos\Theta)$. On the other hand, it is proved that $\sum S_n^{jk} e^{in\theta}$ $(j \neq k)$ is continuous. In this circumstance, the theory due to the author [1], [2], is applicable to the eq.s (17), and on solving for them, we have,

(25)
$$u_1(r,\theta) - u_1^*(r,\theta) = \sum \frac{1}{\Delta_n}\left\{\Gamma_n(a_2, b_2)\mathfrak{A}_{an} + \frac{2i}{\pi a_2}\eta_2 H_n(b_3)\mathfrak{A}_{bn} + U_{1n}^*\right\} J_n(r_1) e^{in\theta},$$
$$u_2(r,\theta) - u_2^*(r,\theta) = \sum \frac{1}{\Delta_n}\left\{J_n(a_1)\Gamma_n(r_2, b_2)\mathfrak{A}_{an} + H_n(b_3)\Lambda_n(r_2, a_2)\mathfrak{A}_{bn} + U_{2n}^*\right\} e^{in\theta},$$
$$u_3(r,\theta) - u_3^*(r,\theta) = \sum \frac{1}{\Delta_n}\left\{\frac{2i}{\pi b_2}\eta_2 J_n(a_1)\mathfrak{A}_{an} + \Lambda_n(b_2, a_2)\mathfrak{A}_{bn} + U_{3n}^*\right\} H_n(r_3) e^{in\theta},$$

which are solutions of Problem E, and

(26)
$$u_1(r,\theta) - u_1^*(r,\theta) = \sum \frac{J_n(r_1)}{J_n'(a_1)}\left\{\frac{\eta_2}{\eta_1}\mathfrak{A}_{an}^c + U_{1n}^*\right\} e^{in\theta},$$
$$u_2(r,\theta) - u_2^*(r,\theta) = \sum \frac{e^{in\theta}}{L_n'(a_2, b_2)}\{L_n(r_2, b_2)\mathfrak{A}_{an}^c - L_n(r_2, a_2)\mathfrak{A}_{bn}^c + U_{2n}^*\},$$
$$u_3(r,\theta) - u_3^*(r,\theta) = \sum \frac{H_n(r_3)}{H_n'(b_3)}\left\{\frac{\eta_2}{\eta_3}\mathfrak{A}_{bn}^c + U_{3n}^*\right\} e^{in\theta},$$

which are the solutions of Problem H, where $\mathfrak{A}_{an}$ and $\mathfrak{A}_{bn}$ are the integrals defined by (15) over the arcs $L_a = L_1$ and $L_b = L_2$ while $\mathfrak{A}_{an}^c$ and $\mathfrak{A}_{bn}^c$ are the integrals over the openings $L_a^c = L_1^c$ and $L_b^c = L_2^c$, respectively. The right hand members of expressions in (25) and (26) represent the secondary fields, in which those terms of $\mathfrak{A}_{an}$, $\mathfrak{A}_{bn}$, $\mathfrak{A}_{an}^c$, and $\mathfrak{A}_{bn}^c$ represent the effects of arcs and openings, respectively, while U_{jn}^* are quantities depending only on the primary fields $u_j^*(r,\theta)$ independently of the arcs and openings which are zero when $u_j^* = 0$. Furthermore, it is shown that $\mathfrak{A}_{an}$, $\mathfrak{A}_{bn}$, $\mathfrak{A}_{an}^c$, $\mathfrak{A}_{bn}^c$, or in general, $\mathfrak{A}_{jn}$, are given by

(27)
$$i\mathfrak{A}_{jn} = \sum_{m=-N}^{N+\nu_j} p_m^j \alpha_{m-n}^j + \begin{cases} \sum_{m=0}^{n} \gamma_{n-m}^j f_{-m}^j, & (n \geqq 0), \\ 0, & (-1 \geqq n \geqq -\nu_j), \\ \sum_{m=n+\nu_j}^{-1} \beta_{n-m}^j f_{-m}^j, & (-\nu_j - 1 \geqq n), \end{cases}$$

where N is a sufficiently large integer. α's, β's, and γ's are known constants relating to the function $X_j(\phi) = \left\{\prod_{m=1}^{2\nu_j}(e^{i\phi} - e^{i\theta_{jm}})\right\}^{-\frac{1}{2}}$, while $\{p_m^j\}$ are known constants which are determined by certain simultaneous linear algebraic equations, and f_{-m}^j are known constants relating to u_j^* [2]. (25) and (26) are the complete formulations of the solutions of Problems E and H, which satisfy all requirements of the problems including the edge conditions (12).

Because it is impossible to describe the details of the discussions including that for the resonances (that is, cases of $\Delta_n=0$ or $J'_n(a_1)=0$ etc.) as well as the detailed expressions of those quantities appeared above, a publication of the present work in its full text is expected in near future in some other journal [3].

References

[1] Hayashi, Y.: A singular integral equation approach to electromagnetic fields for circular boundaries with slots. Jour. Appl. Sci. Res., B, **12**, 331-359 (1965-66).

[2] ——: On the first kind integral equations of Fredholm whose kernel has a log singularity (to appear).

[3] ——: Theory of electromagnetic fields in media bounded by coaxial circular cylinders with slots (to appear).